INTRODUCTION TO GROUP REPRESENTATIONS
GLOSSARY

Sections

Advanced Linear Algebra

Cayley-Hamilton Theorem: If $p_A(x) = det(xI - A)$ is the characteristic polynomial of the square matrix A, then $P_A(A) = 0$ as a matrix polynomial.

Direct Sum: A vector space V can be represented as a direct sum of two subspaces W and W', denoted $V = W + W'$, if each $v = v_1 + v_2$ uniquely with v_1 in W and v_2 in W'. Equivalently, if we choose bases B and B' for W and W' respectively, then $B \cup B'$ is a basis B^* for V. If both W and W' are invariant under A, then the associated matrix for A with respect to B^* is block diagonal $A_{B^*} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$.

Dual Vector Space: The vector space of all linear functionals on V, denoted V^*. Dual representations are also called contragredient.

Invariant Subspace: A subspace W is called **invariant** under A if Av is in W for all w in W. This means that the restriction $A : W \to W$ is defined. Eigenspaces are a special case of invariant subspaces. If we choose a basis for W and complete to a full basis, the associated matrix for A in this basis is block upper-triangular $A_B = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix}$.

Linear Functional: A linear functional on V is a linear transformation $f : V \to \mathbb{C}$. The set of linear functionals on V is called the dual space of V, denoted V^*, and V^* admits a vector space structure in a natural manner. If V admits an inner product, then every linear functional on V can be written in the form $f_v(w) = \langle w, v \rangle$. In addition V^* admits an inner product defined by

$$\langle f_v, f_w \rangle_* = \langle w, v \rangle = \overline{\langle v, w \rangle}.$$

Linear Transformation Space $Hom_{\mathbb{C}}(V, W)$**:** The vector space of linear transformations $T : V \to W$. When $W = \mathbb{C}$, we have V^*, the dual space of V. If we choose bases for

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V^* and W, then every linear transformation T from V to W can be expressed uniquely as

$$T(v) = \sum_{i,j} c_{i,j} v_j^*(v) w_i.$$

In turn, we may associate T with the tensor $t(T) = \sum_{i,j} c_{i,j}(v_j^* \otimes w_i)$ in $V^* \otimes W$.

$Hom_{\mathbb{C}}(V, W)$ admits an inner product defined by $\langle T_1, T_2 \rangle = Trace(T_1 T_2^*)$. If V and W are inner product spaces and we choose orthonormal bases above, we have $\langle T, T \rangle = \sum_{i,j} |c_{i,j}|^2$.

Minimal Polynomial: For a square matrix A, the monic polynomial m_A of smallest degree such that $m_A(A) = 0$ as a matrix polynomial. m_A divides p_A, the characteristic polynomial, and m_A and p_A have the same irreducible factors. If $q(A) = 0$, then m_A divides q. A useful result is that A is diagonalizable if and only if m_A factors into distinct linear factors.

Permutation Matrix: A matrix P_σ of zeros except for exactly one 1 in each row and column. To construct these, let σ be a permutation of $\{1, \ldots, n\}$. We define a linear transformation $T_\sigma : \mathbb{C}^n \to \mathbb{C}^n$ by linearly extending $T_\sigma e_i = e_{\sigma i}$. With respect to the standard basis B, the associated matrix $P_\sigma = [T_\sigma]_B$ is a rearrangement of the columns of the identity matrix. Each permutation matrix P_σ is orthogonal/unitary: $P_\sigma P_\sigma^T = I$. Also $P_\sigma^N = I$ for some positive integer N, so each permutation matrix is diagonalizable over $\mathbb{C}$. The trace of a permutation matrix P_σ is equal to the number of labels in $\{1, \ldots, n\}$ fixed by σ.

Sesquilinear Pairing/Form: Generalizes Hermitian inner product. A sesquilinear pairing $f : V \times V' \to \mathbb{C}$ satisfies the following three conditions:

(1) $f(cv, v') = cf(v, v') = f(v, \bar{c}v')$,
(2) $f(v + w, v') = f(v, v') + f(w, v')$, and
(3) $f(v, v' + w') = f(v, v') + f(v, w')$.

The set of sesquilinear pairings admits a vector space structure with scalar multiplication $(cf)(v, w) = c(f(v, w))$ and addition $(f + h)(v, w) = f(v, w) + h(v, w)$. If $V = V'$, then f is called a sesquilinear form, and if $f(v, w) = \overline{f(w, v)}$, then f is called Hermitian. If $f(v, v) \geq 0$ and $= 0$ if and only if $v = 0$, then f is called a Hermitian inner product on V.

Simultaneous Diagonalization: A set of diagonalizable matrices A_α can be diagonalized using the same basis of eigenvectors if and only if $\{A_\alpha\}$ is a commuting set of matrices. That is, $A_\alpha A_\beta = A_\beta A_\alpha$ for all α, β.

Tensor Product: Denoted by $V \otimes W$. A way to "multiply" vector spaces. In general, a tensor is expressed as a sum of monomial tensors $v \otimes w$. If $B = \{v_i\}$ and $C = \{w_j\}$ are bases for V and W respectively, then a tensor may be expressed uniquely in the form $t = \sum_{i,j} c_{i,j}(v_i \otimes w_j)$. That is, $\{v_i \otimes w_j\}$ is a basis for $V \otimes W$, which in turn has dimension $\dim(V) \cdot \dim(W)$.

We manipulate tensors with the following bilinearity rules:

(1) $c(v \otimes w) = (cv) \otimes w = v \otimes (cw)$,
(2) $(v + v') \otimes w = v \otimes w + v' \otimes w$, and
(3) $v \otimes (w + w') = v \otimes w + v \otimes w'$.

These allow us to switch to expressions in other bases. In general, tensors are defined using universal mapping properties. Here we work only with concrete examples.

Trace: The trace of a square matrix A is the sum of its diagonal entries. In turn, it is also equal to the sum of its eigenvalues, counting multiplicities. We have that

$$Trace(AB) = Trace(BA),$$

so that

$$Trace(PAP^{-1}) = Trace(A).$$

That is, if A is the matrix associated to a linear transformation T, then the trace is unaffected by change of basis, and thus $Trace(T)$ is well-defined. If

$$p(x) = x^n + a_{n-1}x^{n-1} + \dots$$

is the characteristic polynomial of A, then $Trace(A) = -a_{n-1}$.

Analysis on Groups

Character: Depends on context. A one-dimensional representation is called a character of G, but it may also refer to the function $\chi_\pi(g) = Trace(\pi(g))$ on G for any representation π. Representations are equivalent if and only if their characters are equal.

Character Table: For an abelian group, each column corresponds to a group element and each row corresponds to a character. Then entries are the values of the characters. The table of entries is square of size $|G|$, and the rows are orthonormal as elements of $L^2(G)$. In addition, the columns of the table are also orthonormal, considered as elements of $L^2(G^*)$.

For nonabelian G, each column corresponds to a conjugacy class of G, indexed by an element in the class and also labeled with the number of elements in the class. The rows correspond to the (trace) characters for each irreducible class of representations of G. The table has size equal to the number of irreducible classes, which is also equal to the number of conjugacy classes of G. Again the rows are orthonormal as elements of $L^2(G)$; in the table, we must weight each sum by the numbers of elements in each class. The columns are also orthogonal using unweighted sums. Here the length of each column is $\frac{|G|}{|C_x|}$, where C_x is the conjugacy class for x and x represents the column.

Class function: A function f in $L^2(G)$ is called a class function if $f(gxg^{-1}) = f(x)$ for all x, g in G. Two orthonormal bases exist for the subspace of class functions on G. First consider the basis $B_1 = \{\sqrt{|C_x|} f_x\}$ where

$$f_x(y) = \sum_{g \in C_x} \delta_g(y) = \begin{cases} 1 & y \in C_x \\ 0 & \text{otherwise} \end{cases}.$$

Here x runs over a set of representatives for each conjugacy class C_x in G. On the other hand, the characters for each irreducible class of representation on G form an orthonormal basis $B_2 = \{\chi_\pi\}$ of $Class(G)$. Thus the number of irreducible classes of representations for G equals the number of conjugacy classes in G.

Convolution: If f_1 and f_2 are in $L^2(G)$, we define the convolution

$$\begin{aligned}(f_1 * f_2)(x) &= \frac{1}{|G|}\sum_{gg'=x} f_1(g)f_2(g') \\ &= \frac{1}{|G|}\sum_{g\in G} f_1(g)f_2(g^{-1}x) \\ &= \frac{1}{|G|}\sum_{g'\in G} f_1(xg'^{-1})f_2(g').\end{aligned}$$

Convolution has better group-theoretic properties than ordinary multiplication of functions and may be interpreted as a version of matrix multiplication for functions. For instance,

$$\phi_{u,v} * \phi_{u',v'} = 0$$

if the matrix coefficients belong to inequivalent irreducible representations, and

$$d_\sigma \phi_{u,v} * \phi_{u',v'} = \langle u, v'\rangle \phi_{u',v}$$

if the matrix coefficients belong to the irreducible representation σ.

Orthogonal projections onto irreducible types in $L^2(G)$ may be defined using convolutions of characters. The projection onto the subrepresentation of σ-types is given by

$$P_\sigma : L^2(G) \to L^2(G), \qquad P_\sigma f = d_\sigma f * \chi_\sigma = d_\sigma \chi_\sigma * f$$

where χ_σ is the character of σ.

Delta Function: Delta functions give a natural basis for $L^2(G)$, but are not well-suited to representation theory. We define

$$\delta_g(x) = \begin{cases} 1 & x = g \\ 0 & \text{otherwise} \end{cases}.$$

An orthonormal basis for $L^2(G)$ is given by $\{\sqrt{|G|}\delta_g\}$.

Fourier transform: Suppose $B = \{v_i\}$ is an orthonormal basis of the inner product space V. Then every v can be represented uniquely in the form

$$v = \sum_i c_i v_i.$$

The Fourier transform associated to B is the map $F_B : V \to \mathbb{C}^n$ defined by

$$F_B(v) = \widehat{v} = (c_1, \ldots, c_n).$$

The associated inverse Fourier transform $F_B^{-1} : \mathbb{C}^n \to V$ is defined as

$$F_B^{-1}(c_1, \ldots, c_n) = \sum_i c_i v_i.$$

Properties of the Fourier Transform are summarized by Fourier's Trick and Parseval's Identity. Fourier transforms are typically used in the context of analysis. For instance, let G be abelian, $V = L^2(G)$ and B is a basis of characters for G. These ideas generalize for non-abelian G.

$\mathbf{L}^2(G)$: "L two of G." The vector space of all complex-valued functions on G. The notation emphasizes that we use the L^2-norm for the inner product:

$$\langle f, h\rangle = \frac{1}{|G|} \sum_{g \in G} f(g)\overline{h(g)}.$$

A main result states that an orthonormal basis of $L^2(G)$ is given in terms of the matrix coefficients of the irreducible representations of G.

Matrix Coefficient: Let (π, V) be a representation of G. If v^* is in V^* and w is in V, then the matrix coefficient function for v^* and w is given by

$$\phi_{v,w}(g) = v^*(\pi(g)w).$$

Alternatively, when π is unitary, we define $\phi_{v,w}(g) = \langle \pi(g)v, w\rangle$ for v, w in V. If $V = \mathbb{C}^n$, then, with respect to the standard inner product,

$$\phi_{e_i,e_j}(g) = \langle \pi(g)e_j, e_i\rangle.$$

If π is also irreducible, $\phi_{v,w}$ may be considered as an element of the irreducible $G\times G$-type $V^* \otimes V$ in $(L\otimes R, L^2(G))$. If $B = \{u_i\}$ is an orthonormal basis for V, then $\{\sqrt{d_\pi}\phi_{u_i,u_j}\}$ is an orthonormal basis for the unique $V^* \otimes V$-type in $L^2(G)$. In fact, every function f in $L^2(G)$ may be expressed as a sum of matrix coefficients of inequivalent,irreducible representations π:

$$f(g) = \sum_\pi c_{ij\pi} \sqrt{d_\pi}\phi_{u_i,u_j}(g),$$

where $c_{ij\pi} = \langle f, \sqrt{d_\pi}\phi_{u_i,u_j}\rangle$.

Peter-Weyl Theorem: Typically refers to the compact case. For G a finite group, we may decompose

$$L^2(G) \cong \sum_\sigma V_\sigma^* \otimes V_\sigma$$

as an orthogonal direct sum of irreducible $G \times G$ subrepresentations under $L \otimes R$. Each irreducible type σ of G occurs exactly once, and passage from right to left is implemented by forming matrix coefficients for each σ.

Planchерel Formula: Let $f^*(x) = \overline{f(x^{-1})}$. If f and h are in $L^2(G)$, then

$$\langle f, h\rangle = \sum_{\sigma} d_\sigma Trace(\sigma(f)\sigma(h)^*),$$

where the sum ranges over representatives σ for each irreducible class.

This can be though of as a version of Parseval's Identity; we obtain a basis-free formula in the non-abelian case. The Schur Orthogonality Relations and Plancherel Formula reconcile three Hermitian inner products invariant under $G \times G$: on the matrix coefficient span in $L^2(G)$ of the irreducible representation (σ, V_σ), on $V_\sigma \otimes V_\sigma^*$, and on $Hom_{\mathbb{C}}(V_\sigma, V_\sigma)$.

Projection Formula onto Irreducible Types: Let (π, V) be a unitary representation of a finite group G. Let σ be an irreducible representation of G. The orthogonal projection onto the space of σ-types in V is given by the formula

$$P_\sigma : V \to V, \qquad P_\sigma v = \pi(d_\sigma \overline{\chi_\sigma})v = \frac{d_\sigma}{|G|}\sum_{g\in G}\overline{\chi_\sigma(g)}\pi(g)v,$$

where χ_σ is the character of σ.

Regular Representation of G: The left and right regular representations of G act on the vector space $L^2(G)$ with actions given by

$$[L(g)f](x) = f(g^{-1}x), \qquad and \qquad [R(g)f](x) = f(xg).$$

One may also consider both simultaneously as a representation of $G \times G$ on $L^2(G)$. Letting (π, V) range over a complete set of inequivalent, irreducible representations of G, $L \otimes R$ decomposes into $G \times G$-types as

$$L^2(G) \cong \sum_{\pi} V^* \otimes V.$$

This equivalence is implemented by passage to matrix coefficients. As a consequence, all irreducible representations of G occur in $L^2(G)$ as subrepresentations.

Schur Orthogonality Relations: Let (π, V) and (π', V') be irreducible unitary representations of a finite group G. If u, v in V, u', v' in V', then

$$\langle \phi_{u,v}, \phi_{u',v'}\rangle_{L^2} = \begin{cases} \frac{1}{d_\pi}\langle u, u'\rangle\overline{\langle v, v'\rangle} & \text{if } V = V', \\ 0 & \text{if } \pi, \pi' \text{ inequivalent} \end{cases}.$$

Weighted Average of a Representation $\pi(f)$: Let (π, V) be a unitary representation of G, and suppose f is an element in $L^2(G)$. We define the weighted average of π by f as

$$\pi(f) : V \to V, \qquad \pi(f)v = \frac{1}{|G|}\sum_{g\in G} f(g)\pi(g)v$$

One may consider the map as an algebra homomorphism

$$\pi(\cdot) : L^2(G) \to Hom_{\mathbb{C}}(V, V),$$

where the multiplication on $L^2(G)$ is convolution. When G is non-abelian, weighted averages of irreducible representations correspond to a version of Fourier transform, as seen in the Plancherel Formula.

Group Theory

Commutator Subgroup: If G is a group, the commutator subgroup $[G, G]$ is generated by all products $xyx^{-1}y^{-1}$ for all x, y in G. It is normal in G, and $G/[G, G]$ is called the abelianization of G. If N is a normal subgroup and G/N is abelian, then $[G, G] \subseteq N$.

Conjugacy Class: If x is in G, the conjugacy class of x is the set $C_x = \{gxg^{-1} \mid g \in G\}$. Central elements form singleton classes, and the set of all conjugacy classes partition G. When G is finite, we have the class equation:

$$|G| = |Z(G)| + \sum |C_x|,$$

where the sum is over classes with more than one element. We further note that $|C_x| = |G|/|C(x)|$, where $C(x)$ is the centralizer of x in G.

Dihedral Group: The symmetry group of a regular n-gon, denoted as either D_{2n} or D_n. Suppose the n-gon is centered at the origin in the plane and vertices are labeled with 1 on the positive x-axis and increasing counter clockwise. Then D_{2n} is generated by c (reflection across the x-axis) and r (rotation counter-clockwise by $\frac{2\pi}{n}$). Then each of the $2n$ elements of D_{2n} may be written in the form r^k (rotations) or cr^k (reflections). Furthermore D_{2n} is described completely by the relations

$$r^n = e, \quad c^2 = e, \quad crc = r^{-1}.$$

One can also describe elements of D_{2n} in cycle notation as a subgroup of the symmetric group S_n.

Fundamental Theorem of Finite Abelian Groups (FTFAG): Every finite abelian group is isomorphic to a product of cyclic groups in the form

$$\mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_k,$$

where n_i divides n_{i+1}.

Group Action: A group G acts on a set X by the action $\pi(g) : X \to X$ if, for all g, h in G and x in X, (1) $\pi(gh)x = \pi(g)[\pi(h)x]$ and (2) $\pi(e)x = x$. In other words, π is a homomorphism from G into the group of bijections on X.

Normal Subgroup: A subgroup N of G such that gng^{-1} is in N for all n in N and g in G. This is equivalent to the condition $gN = Ng$ for all g in G. Normality allows us to put a group structure on the quotient G/N; that is, we have the coset equality $(xN)(yN) = xyN$ for all x and y in G.

Orbit: If π is a group action of G on a set X, then the **orbit** of x in X under G is the subset $\{\pi(g)x \mid g \in G\}$. The orbit of x consists of all points in X that can be reached from

x using an element of G. The orbits partition X. If G and X are finite, then the orbit of x has $|G|/|Stab_G(x)|$ elements, where $Stab_G(x)$ is the stabilizer subgroup of x in G.

Quaternion Group: This group is non-abelian with eight elements. As a set, $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, and the multiplication is given by

$$i^2 = j^2 = k^2 = -1, \quad ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik.$$

Every subgroup of Q is normal.

Quotient Group/Map: If N is a normal subgroup, then the coset space admits a group structure with multiplication $(xN)(yN) = xyN$. We can extend any representation on G/N to G using the quotient map $q : G \to G/N$ defined by sending g to gN.

Stabilizer Subgroup: If π is a group action of G on the set X, then the **stabilizer subgroup** of x under G, denoted $Stab_G(x)$, is the subgroup of all g in G such that $\pi(g)x = x$.

Transitive: A group action is called **transitive** if there is only one orbit. That is, if x and y are in X, then there exists a g in G such that $\pi(g)x = y$.

Linear Algebra

Associated Matrix: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $T(v) = Av$ for an $m \times n$ matrix $A = [T(e_1) \dots T(e_n)]$. We call this A the associated matrix for T with respect to the standard bases for $\mathbb{R}^m$ and $\mathbb{R}^n$. This definition also applies to the complex case.

Basis: A **basis** of a vector space V is a set of vectors $B = \{v_1, \dots, v_n\}$ such that every vector v in V can be expressed uniquely as a linear combination $v = c_1v_1 + \dots c_nv_n$. A basis is called **orthonormal** if the basis vectors are unit and mutually orthogonal.

Change of Basis: If we change basis from $B = [v_1 \dots v_n]$ to $C = [w_1, \dots, w_n]$, the coordinates of a vector v in V change. If $V = \mathbb{R}^n$ or $\mathbb{C}^n$, we define the matrix $P_B = [v_1 \dots v_n]$, which carries the coordinate vector $[v]_B$ to v. We change from B-coordinates to C-coordinates by applying the matrix $P_{C \leftarrow B} = P_C^{-1} P_B$. On associated matrices, we change basis by conjugation, $A_C = P_{C \leftarrow B} A_B P_{C \leftarrow B}^{-1}$.

Characteristic Equation: $Av = cv$. Here A is a square matrix, v is a vector, and c is a scalar. This equation is used to find eigenvalues and eigenvectors for A. Geometrically, it states that the line through v and the origin is mapped back into itself by A. This line is a special case of an subspace invariant under A.

Characteristic Polynomial: For a square matrix A, the characteristic polynomial of A is defined as $p_A(x) = det(xI - A)$. Its roots are the eigenvalues of A. . Suppose

$$p_A(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0.$$

Since $p_A(x)$ is unaffected by replacing A with PAP^{-1}, the a_i are also unaffected. Note that $a_{n-1} = -Trace(A)$ and $a_0 = (-1)^n det(A)$. In addition, this means one may define $p_T(x)$ for a linear transformations $T : V \to V$.

Coordinate vector: If $B = \{v_1, \ldots, v_n\}$ is a basis for the vector space V, then the **coordinate vector** for $v = c_1 v_1 + \ldots c_n v_n$ is the column vector $[v]_B = (c_1, \ldots c_n)$ in F^n, where F is $\mathbb{R}$ or $\mathbb{C}$.

Diagonal Form: If there exists a basis of eigenvectors for the square matrix A, then A can be put into diagonal form; that is there exists a change of basis such that $P^{-1}AP = D$ for a diagonal matrix D. In this case, $P = [v_1 \ldots v_n]$ using the eigenvector basis and D is the diagonal matrix with eigenvalues in the same order as the v's.

Eigenvalue/Eigenvector: If v is a solution to the characteristic equation $Av = cv$, then v is called an eigenvector for A with eigenvalue c. The span of an eigenvector is a line mapped back into itself by A. The eigenvalue is the scaling factor of A on the line.

Eigenspace: The eigenspace for A with eigenvalue c is the subspace of all eigenvectors v with eigenvalue c; that is, all solutions to $Av = cv$. This is the null space $Null(A - cI)$.

Fourier Trick/Parseval's Identity: If we have an orthonormal basis $B = \{v_1, \ldots, v_n\}$ and $v = c_1 v_1 + \cdots + c_n v_n$, then $c_i = \langle v, v_i \rangle$. Also $||v||^2 = \sum_i |c_i|^2$.

Gram-Schmidt Orthogonalization: For inner product spaces, a process for turning a basis into an orthonormal basis. If $B = \{v_i\}$, we proceed inductively. First let $u_1 = v_1/||v_1||$. Then if we have the orthonormal set $\{u_i\}$ $(1 \le i \le k)$, we define

$$u'_{k+1} = v_{k+1} - \sum \langle v_{k+1}, u_i \rangle u_i, \quad \text{and} \quad u_{k+1} = u'_{k+1}/||u'_{k+1}||.$$

Hermitian matrix: A complex $n \times n$ matrix X such that $X^* = X$. By the Spectral Theorem, a Hermitian matrix X can always be diagonalized using an orthonormal basis of eigenvectors; that is, $D = W^T X W$ for some diagonal matrix D with real entries and some unitary matrix W.

Inner Product/Hermitian Inner Product: A function of two variables on a vector space that measures lengths of vectors and angles between vectors. For $\mathbb{R}^2$ and $\mathbb{R}^3$, one has the usual dot product. On complex vector spaces, we have Hermitian inner products. Instead of the symmetric property, we have the Hermitian property $\langle v, w \rangle = \overline{\langle w, v \rangle}$.

Orthogonal: Two vectors v, w are called **orthogonal** if $\langle v, w \rangle = 0$. Subspaces W and W' and are called orthogonal if $\langle v, w \rangle = 0$ for all v in W and w in W'.

Orthogonal Complement: If W is a subspace of the inner product space V, then $W^\perp$, the orthogonal complement of W in V, is defined as the set of all v in V such that $\langle v, w \rangle = 0$ for all w in W. Then we can write V as an orthogonal direct sum $V = W \oplus W^\perp$. Also if X is symmetric or orthogonal, then W is invariant under X if and only if $W^\perp$ is invariant under X. For complex inner product spaces, replace "orthogonal" with "unitary" and "symmetric" with "Hermitian."

Orthogonal Direct Sum: In an inner product space, a direct sum $V = W \oplus W'$ in which the vectors in W are orthogonal to the vectors in W'.

Orthogonal Matrix: A real $n \times n$ matrix P such that $PP^T = P^T P = I$. Equivalently, the rows and columns form orthonormal bases of $\mathbb{R}^n$. Equivalently, P acts by isometries on $\mathbb{R}^n$: $\langle Pv, Pw \rangle = \langle v, w \rangle$.

Orthogonal Projection: If W is a subspace of the inner product space V, then each v in V can be written uniquely as $v = v_1 + v_2$ for v_1 in W and v_2 in $W^\perp$. We define the **orthogonal projection** P_W onto W by $P_W(v) = v_1$. The following properties hold:

$$P_W^2 = P_W, \quad P_W + P_{W^\perp} = I, \quad P^T = P, \quad P_W P_{W^\perp} = P_{W^\perp} P_W = 0.$$

For complex vector spaces, replace P^T with P^*.

Orthonormal Basis: A basis is called **orthonormal** if the basis vectors are unit and mutually orthogonal.

Spectral Theorem: For real vector spaces, any real symmetric matrix S can be diagonalized by an orthogonal matrix; in other words, there exists an orthonormal basis of eigenvectors with real eigenvalues. For complex vector spaces, any Hermitian matrix can be diagonalized using a unitary matrix. One interpretation is that a Hermitian operator X can be written as a sum $X = \sum_i \lambda_i P_i$, where the λ_i are the (real) eigenvalues of X and the P_i are the orthogonal projections onto the eigenspaces.

Symmetric Matrix: A $n \times n$ matrix S such that $S^T = S$. By the Spectral Theorem, a real symmetric matrix S can always be diagonalized using an orthonormal basis of eigenvectors; that is, $D = P^T S P$ for some diagonal matrix D with real entries and some orthogonal matrix P.

Unit Vector: A vector v such that $||v|| = 1$.

Unitary Matrix: A complex $n \times n$ matrix such that $UU^* = U^*U = I$. Equivalently, the rows and columns form orthonormal bases of $\mathbb{C}^n$. Equivalently, using the standard Hermitian inner product, U acts by isometries on $\mathbb{C}^n$: $\langle Uv, Uw \rangle = \langle v, w \rangle$ for all v, w in $\mathbb{C}^n$. A unitary matrix U can always be diagonalized using an orthonormal basis of eigenvectors; that is, $D = W^*UW$ for some diagonal matrix D with unit diagonal entries and some unitary matrix W. This fact is analogous to the Spectral Theorem for Hermitian matrices:

$$U = \sum_\lambda \lambda P_\lambda,$$

where $\lambda = e^{i\theta}$ ranges over the eigenvalues of U and P_λ are the orthogonal projections onto the eigenspaces.

Representation Theory

Equivalence: If (π, V) and (π', V') are representations of G, then a linear isomorphism $L : V \to V'$ is called an **equivalence** if $\pi'(g)L = L\pi(g)$ for all g in G. Essentially, π and

π' are the "same" representation, just with different labels. A special case of "intertwining operator."

Faithful: A group action (or representation) π is called **faithful** if $\pi(g) = I$ implies $g = e$. If we consider π as a homomorphism from G into the group of bijections on X, then $Ker(\pi)$ is trivial.

Fully Reducible: A representation (π, V) is called **fully reducible** if V decomposes into an (orthogonal) direct sum of irreducible subrepresentations. This is analogous to diagonalizing a unitary matrix using an orthonormal basis; fixing an orthonormal basis, each $\pi(g)$ can be made block diagonal with unitary blocks simultaneously. Over $\mathbb{C}$, finite-dimensional representations of finite groups are fully reducible.

To determine the multiplicity of an irreducible type σ in π, one calculates $n_\sigma = \langle \chi_\pi, \chi_\sigma \rangle$ with respect to the L^2-norm. Each χ is the character of the corresponding representation.

Intertwining Operator: If (π, V) and (π', V') are representations of G, then a linear transformation $L : V \to V'$ is called **intertwining operator** (or G-map, or G-equivariant map) if $\pi'(g)L = L\pi(g)$ for all g in G. Essentially, L preserves the action of G. If L is an isomorphism, we call L an equivalence. Then π and π are the "same" representation, just labeled differently. We denote the vector space of intertwining operators by $Hom_G(V, V')$.

Invariant Hermitian Inner Product: The inner product for a unitary representation (π, V). Invariant means $\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle$ for all v, w in V and g in G. With respect to an orthonormal basis, each $\pi(g)$ is a unitary matrix. For representations of finite groups, every inner product can be made invariant by an averaging operation. That is, every representation can be made unitary.

Irreducible Representation: A representation (π, V) of G is called **irreducible** if 0 and V are the only subrepresentations. One-dimensional representations are automatically irreducible.

Reducible Representation: A representation (π, V) that contains a subrepresentation (π, W) with W not equal to either 0 or V.

Representation: A representation (π, V) of the group G is a homomorphism $\pi : G \to GL(V)$, the group of invertible linear transformations on V. Alternatively, π is a group action on V by linear transformations.

Schur's Lemma: Let (π, V) and (π', V') be irreducible representations of G. Then any intertwining operator $L : V \to V'$ is an equivalence or zero. If an equivalence, L is unique up to scalar. For irreducible unitary representations, this implies that invariant Hermitian inner products are unique up to real positive scalar, and that there exists no nonzero invariant sesquilinear pairings between inequivalent irreducible representations.

Subrepresentation: Suppose W is a subspace of V. A subrepresentation is a representation (π, W) within a representation (π, V). That is, for all g in G and w in W, $\pi(g)w$ is in W. For a unitary representation, $(\pi, W^\perp)$ is also a subrepresentation of (π, V).

Type: Let (σ, W) be an irreducible representation of G. If (π, V') is an irreducible subrepresentation of (π, V), we say V' has type σ if V' and W are equivalent. In other words, V' and W belong to the same equivalence class under equivalence of representations. We say V' has irreducible type σ.

Unitary Equivalence: An equivalence of unitary representations (π, V), (π', V') implemented by a unitary transformation. That is, L also satisfies $\langle Lv, Lw\rangle' = \langle v, w\rangle$ for all v, w in V.

Unitary Representation: A representation (π, V) such that each $\pi(g)$ preserves a Hermitian inner product $\langle \cdot, \cdot\rangle$ is called **unitary**. That is, $\langle \pi(g)v, \pi(g)w\rangle = \langle v, w\rangle$ for all v, w in V and g in G. With respect to an orthonormal basis B, each $[\pi(g)]_B$ is a unitary matrix. Unitary representations on finite-dimensional vector spaces are fully reducible.