INTRODUCTION TO GROUP REPRESENTATIONS
JULY 1, 2012
LINEAR ALGEBRA REVIEW 3

Here we describe some of the main linear algebra constructions used in representation theory. Since we are working with finite-dimensional vector spaces, a choice of basis identifies each new space with some $\mathbb{C}^n$. What makes these interesting is how they behave with respect to linear transformation, which in turn gives ways to construct new representations from old ones.

Let V and W be finite-dimensional vector spaces over $\mathbb{C}$, and choose bases $B = \{v_1, \ldots, v_m\}$ for V and $C = \{w_1, \ldots, w_n\}$ for W.

Direct Sum:

There are two types of direct sums, internal and external. The internal direct sum expresses a given vector space V' in terms of two (or more) subspaces V and W.

Definition: We say V' is the **internal direct sum** of subspaces V and W, written $V' = V + W$, if each vector in V' can be written uniquely as a sum $v' = v + w$ with v in V and w in W. Equivalently, this condition holds if and only if $B \cup C$ is a basis for V'. In turn, we also have this condition if $dim(V) + dim(W) = dim(V')$ and $V \cap W = \{0\}$.

In the special case where V' admits an inner product and $W = V^{\perp}$, we call $V' = V \oplus W$ an **orthogonal direct sum**. If B and C are orthonormal bases, then $B \cup C$ is an orthonormal basis for V'.

On the other hand, the main idea here can be applied to combine two known vector spaces.

Definition: The **external direct sum** of V and W, also written $V' = V \oplus W$, is first defined as the set of all ordered pairs (v, w) with v in V and w in W. Scalar multiplication is defined by $c(v, w) = (cv, cw)$, and addition is defined by $(v, w)+(v', w') = (v+v', w+w')$. One checks the other axioms for a vector space.

Note that the external direct sum of V and W can be expressed as the internal direct sum of $(V, 0)$ and $(0, W)$. A basis for $V \oplus W$ is given by $\{(v_i, 0)\} \cup \{(0, w_j)\}$.

If $S : V \to V'$ and $T : W \to W'$ are linear transformations, we obtain a linear transformation $(S, T) : V \oplus W \to V' \oplus W'$ by $(S, T)(v, w) = (Sv, Tw)$. The reader should verify that (S, T) is linear. This definition allows a natural method for constructing direct sums of representations.

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Dual Vector Spaces:

Definition: A **linear functional** on V is a linear transformation $f : V \to \mathbb{C}$, and the vector space of all linear functionals on V is called the **dual vector space** V^*.

If V also admits an inner product, then one may describe each linear functional f uniquely in the form $f_v(v') = \langle v', v\rangle$ for some v in V. In fact, one may choose any nonzero v in $(Ker f)^\perp$ and rescale to match values to f. In turn, V^* admits an inner product defined by

$$\langle f_v, f_w\rangle_* = \langle w, v\rangle = \overline{\langle v, w\rangle}.$$

If $T : V \to W$ is a linear transformation, then we have an induced linear transformation $T^* : W^* \to V^*$ by defining $(T^*w^*)(v) = w^*(Tv)$. If V and W admit Hermitian inner products,

$$(T^*f)(v) = f(Tv) = \langle Tv, w\rangle = \langle v, T^*w\rangle,$$

where the latter T^* represents the adjoint of T with respect to the inner product. If we are working in coordinate spaces $\mathbb{C}^k$, then T^* is conjugate transpose of matrices:

$$w^*Tv = w^*(T^*)^*v = (T^*w)^*v.$$

Suppose $S : W \to X$, so that $ST : V \to X$. If f is in X^* then

$$[(ST)^*f](v) = f(STv) = (S^*f)(Tv) = [T^*(S^*f)](v).$$

Thus $(ST)^* = T^*S^*$. For representations, if π^* is to be a group action on V^*, we need $\pi^*(gh) = \pi^*(g)\pi^*(h)$. We correct the order using inverses.

Definition: The dual basis B^* for V^* with respect to B is the set $\{v_1^*, \dots, v_m^*\}$, where each v_i^* is defined by

$$v_i^*(v_i) = 1, \quad \text{and} \quad v_j^*(v_i) = 0 \ \text{ otherwise.}$$

If we have an inner product space with orthonormal basis B, then the corresponding dual basis has vectors $v_i^* = \langle \cdot, v_i\rangle$. Note that $dim(V) = dim(V^*)$.

If $T : V \to W$ is an invertible linear transformation, then $\{Tv_1, \dots, Tv_n\}$ is a basis for W with corresponding dual basis $\{(Tv_1)^*, \dots, (Tv_n)^*\}$, where $(Tv_i)^*(w) = v_i^*(T^{-1}w)$. Note that

$$(Tv_i)^*(Tv_j) = v_i^*(T^{-1}Tv_j) = v_i^*(v_j),$$

confirming the dual basis property.

One way to interpret: Suppose we wish to replace V with an isomorphic vector space W. To convert from linear functionals on V to linear functionals on W, we send v^* to $(T^{-1})^*v^*$. For representations, $W = V$ and $T = \pi(g)$. That is, we define the group action by $[\pi^*(g)v^*](v') = v^*(\pi(g)^{-1}v')$.

Another way to interpret: A group action expresses symmetries on an object X; that is, the group action leaves some quality of X unchanged. If $T : V \to V$ is an isomorphism, then the basis $B = \{v_i\}$ is carried to the basis $B_T = \{Tv_i\}$. Note that the definition of $v^*(v')$ requires no basis; that is, this quantity remains unchanged no matter how we pass to coordinates. So if we change V by an isomorphism T, interpreted as a change of basis, the dual basis changes by $(T^{-1})^*$, and $v'(v)$ is unchanged. For representations, we again arrive at the group action on V^*.

Example: Consider the linear functional $f : \mathbb{C}^2 \to \mathbb{C}$ defined by $f(x, y, z) = x - y$. Then

$$f(x, y, z) = [1 - 1] \begin{bmatrix} x \\ y \end{bmatrix}.$$

If $T : \mathbb{C}^3 \to \mathbb{C}^2$ is represented by the matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix}$, then

$$(T^* f)(x, y, z) = [1 - 1]A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -x + y - 4z.$$

Linear Transformation Spaces:

Definition: The set of all linear transformations $T : V \to W$ is denoted by $Hom_{\mathbb{C}}(V, W)$. As a vector space, we define scalar multiplication by $(cT)(v) = c(T(v))$ and addition by $(S + T)(v) = (Sv) + (Tv)$. One verifies the other axioms for a vector space.

With respect to choice of bases B and C, we can identify each T with an $n \times m$ matrix M_T such that $[Tv]_C = M_T[v]_B$. This identification yields an isomorphism of $Hom_{\mathbb{C}}(V, W)$ with the matrix vector space $M(n, m, \mathbb{C})$.

In turn, $Hom_{\mathbb{C}}(V, W)$ admits a Hermitian inner product defined by $\langle T_1, T_2 \rangle = Trace(T_1 T_2^*)$. With a choice of orthonormal bases for V and W, the norm squared of T_1 with associated matrix $M_{T_1} = [c_{i,j}]$ equals $\sum |c_{i,j}|^2$.

On the other hand, each linear functional is a linear transformation into $\mathbb{C}$, and we have seen how to identify V^* with $M(1, m, \mathbb{C})$, the space of row vectors. Now every element of $Hom_{\mathbb{C}}(V, W)$ may be written uniquely in the form

$$T(v) = \sum_{i,j} c_{i,j} v_j^*(v) w_i.$$

Again T is identified with the $n \times m$ matrix $[c_{i,j}]$. Thus we see that $\{T_{i,j}\}$ is a basis for $Hom_{\mathbb{C}}(V, W)$, where $T_{i,j}(v) = v_j^*(v)w_i$, and $dim(Hom_{\mathbb{C}}(V, W)) = mn$.

Now suppose $T_1 : V' \to V$ and $T_2 : W \to W'$ are linear transformations. Then $T_2 T T_1$ is an element of $Hom_{\mathbb{C}}(V', W')$, and, after choosing bases, the associated matrix is $M_{T_2} M_T M_{T_1}$.

Suppose we wish to replace V and W with isomorphic vector spaces V' and W' using $T_1 : V \to V'$ and $T_2 : W \to W'$. Then we replace the element T in $Hom(V, W)$ with $T_2 T T_1^{-1}$ in $Hom_{\mathbb{C}}(V', W')$. For representations, $V' = V$, $W' = W$, $T_1 = \pi(g)$ acts on V, and $T_2 = \pi'(g)$ acts on W to give $\sigma(g)T = \pi'(g)T\pi(g)^{-1}$. For the dual space V^*, we use the trivial action on $W = \mathbb{C}$, so $\pi' = I$.

Example: Let $T : \mathbb{C}^3 \to \mathbb{C}^2$ be given by $T(x, y, z) = (x + y - z, 2x + 3y)$. Suppose

$$T_1 : \mathbb{C}^3 \to \mathbb{C}^3 \quad \text{is given by} \quad T_1(x, y, z) = (3x + y + z, 2x + y, -y + 2z)$$

and

$$T_2 : \mathbb{C}^2 \to \mathbb{C}^2 \quad \text{is given by} \quad T_2(x, y) = (x + y, 2x - 3y).$$

Then

$$\begin{aligned} T_2TT_1(x, y, z) &= \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= (-17x + 8y + z, -26x - 9y - 8z). \end{aligned}$$

Tensor Products:

If the direct sum is thought of as a means to add vector spaces, then tensoring is the way we multiply vector spaces. We will take a somewhat naive approach here, as our focus is working with concrete examples of representations. In general, tensors are defined in terms of universal mapping properties without reference to coordinates. Tensors are the natural objects for moving from linearity to multilinearity.

Definition: With our choices of bases B and C for V and W, we form the set $B \otimes C = \{v_i \otimes w_j\}$. The set $V \otimes W$ is defined as the set of all linear combinations of elements in $B \otimes C$. That is, a tensor in $V \otimes W$, the **tensor product** of V and W, is an element

$$t = \sum_{i,j} c_{i,j}(v_i \otimes w_j),$$

and the vector space of tensors has dimension mn. For any v in V and w in W, the monomial tensor $v \otimes w$ is defined in terms of the basis by applying the following relations:

(1) $c(v \otimes w) = (cv) \otimes w) = v \otimes (cw)$,
(2) $(v + v') \otimes w = v \otimes w + v' \otimes w$, and
(3) $v \otimes (w + w') = v \otimes w + v \otimes w'$.

Often it is sufficient to work with monomials and check that results extend linearly to general tensors. If we choose new bases B' and C', then the bilinearity conditions above allow us to convert from linear combinations in $B \otimes C$ to linear combinations in $B' \otimes C'$. As with the direct product, if we have linear transformations $T_1 : V \to V'$ and $T_2 : W \to W'$, then the map

$$\sum c_{i,j}(v_i \otimes w_j) \mapsto \sum c_{i,j}(T_1 v_i \otimes T_2 w_j)$$

sends tensors in $V \otimes W$ to $V' \otimes W'$. For representations, we use

$$(\pi \otimes \pi')(g)(v \otimes w) = \pi(g)v \otimes \pi'(g)w.$$

An important case of a tensor product is $Hom_{\mathbb{C}}(V, W)$. In this case, we define an isomorphism

$$i : V^* \otimes W \to Hom_{\mathbb{C}}(V, W) \quad \text{by} \quad \sum c_{i,j}(v_j^* \otimes w_i) \mapsto T(v) = \sum c_{i,j} v_j^*(v) w_i.$$

In fact, if we replace V and W with isomorphic vector spaces V' and W' using $T_1 : V \to V'$ and $T_2 : W \to W'$, then the corresponding transforms factor through i:

$$\sum c_{i,j}(T_1^{-1} \otimes T_2)(v_j^* \otimes w_i) = \sum c_{i,j}((T_1^{-1})^* v_j^* \otimes T_2 w_i)$$

is carried to $T_2 T T_1^{-1}$.

Example: Suppose B is the standard basis of $\mathbb{C}^2$, and suppose

$$v_1 = (1, -1) \quad \text{and} \quad v_2 = (-1, 2).$$

Then

$$\begin{aligned} v_1 \otimes v_2 &= (e_1 - e_2) \otimes (-e_1 + 2e_2) \\ &= e_1 \otimes (-e_1 + 2e_2) - e_2 \otimes (-e_1 + 2e_2) \\ &= -e_1 \otimes e_1 + 2e_1 \otimes e_2 + e_2 \otimes e_1 - 2e_2 \otimes e_2. \end{aligned}$$

On the other hand, $e_1 = 2v_1 + v_2$, and $e_2 = v_1 + v_2$. Thus

$$\begin{aligned} e_1 \otimes e_2 &= (2v_1 + v_2) \otimes (v_1 + v_2) \\ &= 2v_1 \otimes (v_1 + v_2) \ + \ v_2 \otimes (v_1 + v_2) \\ &= 2v_1 \otimes v_1 \ + \ 2v_1 \otimes v_2 \ + \ v_2 \otimes v_1 \ + \ v_2 \otimes v_2. \end{aligned}$$

Let $T_1(x, y) = (x + y, 2x + y)$ and $T_2(x, y) = (-x - y, -x + y)$. Then

$$\begin{aligned} (T_1 \otimes T_2)(e_1 \otimes e_2) &= T_1 e_1 \otimes T_2 e_2 = (e_1 + 2e_2) \otimes (-e_1 + e_2) \\ &= -e_1 \otimes e_1 \ - \ 2e_2 \otimes e_1 \ + \ e_1 \otimes e_2 \ + \ 2e_2 \otimes e_2. \end{aligned}$$