INTRODUCTION TO GROUP REPRESENTATIONS
MAIN RESULTS FOR ANALYSIS ON GROUPS
JULY 23, 2012

Finite Abelian Groups:

Let G be a finite group. Then every irreducible representation of G has dimension one, and the set of all such representations (called characters) forms a group G^* under multiplication of values.

Row Orthogonality of Characters: Let χ and χ' be characters of G. Then

$$\frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\chi'(g)} = \begin{cases} 1 & \chi = \chi' \\ 0 & \text{otherwise} \end{cases}.$$

To set up the character table for G, we index the columns by elements of G and the rows by the characters of G. The rows of G are orthogonal with respect to the L^2-norm, and it follows that the columns are also orthogonal.

Column Orthogonality of Characters: Let (π, V) and (π', V') be irreducible, unitary representations of G. Then

$$\frac{1}{|G|}\sum_{\chi\in G^*}\chi(g)\chi'(h) = \begin{cases} 1 & g = h \\ 0 & \text{otherwise} \end{cases}.$$

Let $L^2(G)$ be the vector space of complex-valued functions on G. Because $|G| = |G^*|$, the characters form an orthonormal basis for $L^2(G)$ using the L^2-norm:

$$\langle f, h\rangle = \frac{1}{|G|} f(g)\overline{h(g)}.$$

Since we have an orthonormal basis, we obtain

Fourier's Trick: For f in $L^2(G)$, we have

$$f(g) = \sum_{\chi\in G^*} c_\chi \chi(g),$$

where $c_\chi = \langle f, \chi\rangle$.

Parseval's Identity: $\langle f, f\rangle = \sum_\chi |c_\chi|^2$.

In terms of representation theory of G, one decomposes $L^2(G)$ as follows:

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Decomposition of $L^2(G)$: Let $V_\chi = \mathbb{C}\chi$. Then G acts by the left regular action L (resp. right regular action R) on V_χ by $\overline{\chi}$ (resp. χ). As representations of G with either action,

$$L^2(G) = \bigoplus_{\chi \in G^*} V_\chi.$$

To find the projection operators onto each space of χ-types, we use convolution:

Projection Formula for Functions: The orthogonal projection $P_\chi : L^2(G) \to V_\chi$ is defined by

$$P_\chi f = f * \chi = \chi * f.$$

Projection Formula for Representations: Let V^χ be the subrepresentation of χ-types in V. The orthogonal projection $P_\chi : L^2(G) \to V^\chi$ is defined by

$$P_\chi v = \pi(\overline{\chi})v,$$

where

$$\pi(f)v = \frac{1}{|G|}\sum_{g \in G} f(g)\pi(g)v.$$

From this we obtain a Spectral Theorem analogue for representations:

$$\pi(g) = \sum_{\chi} \chi P_\chi.$$

Once we classify all characters of G, we have a method for counting the irreducible types in any finite-dimensional representation (π, V) of G. Because G is finite abelian, it is possible to find a joint eigenbasis B of V such that $[\pi(g)]_B$ is diagonal for all g in G. Taking traces,

Multiplicity Formula: The multiplicity of χ in π is given by $n_\chi = \langle \chi_\pi, \chi \rangle$.

General Finite Groups:

Character Table for Finite G: The columns are indexed by the conjugacy classes C_x in G, indexed by a representative and the number of elements in the class. The rows are indexed by the characters of the irreducible representations of G, with a representative chosen for each class. Since the then number of irreducible classes equals the number of conjugacy classes, the table is square. Since the characters form an orthonormal basis of $Class(G)$, the rows are orthonormal, weighted by orders of conjugacy classes:

$$\frac{1}{|G|}\sum_{g \in G} \chi_\pi(g)\overline{\chi_{\pi'}(g)} = \begin{cases} 1 & \pi \cong \pi' \\ 0 & \text{otherwise} \end{cases}.$$

The columns are also orthogonal (unweighted):

$$\frac{1}{|G|}\sum_{\pi\in\widehat{G}}\chi_\pi(g)\overline{\chi_\pi(h)} = \begin{cases}\frac{1}{|C_x|} & g=h\\ 0 & \text{otherwise}\end{cases}.$$

For analysis on G, consider the matrix coefficient map $\phi : V^* \otimes V \to L^2(G)$ defined by linearly extending

$$\langle\cdot, v\rangle \otimes u \mapsto \phi_{u,v}(g) = \langle \pi(g)u, v\rangle.$$

Schur Orthogonality Relations: Let (π, V) and (π', V') be irreducible, unitary representations of G. Suppose u, v in V and u', v' in V'. Then

$$\frac{1}{|G|}\sum_{g\in G}\pi_{u,v}(g)\overline{\phi_{u',v'}(g)} = \begin{cases}\frac{1}{d_\pi}\langle u, u'\rangle\overline{\langle v, v'\rangle} & \pi=\pi'\\ 0 & \pi,\pi' \text{ inequivalent}\end{cases}$$

Peter-Weyl Theorem (General Finite Case): As a unitary representation of $G \times G$, $(L \otimes R, L^2(G))$ decomposes into irreducibles as

$$L^2(G) = \oplus_\sigma V_\sigma^* \otimes V_\sigma,$$

where (σ, V_σ) ranges over a complete set of inequivalent, irreducible representations of G.

If we choose an orthonormal basis $\{u_i\}$ for V_σ, then $\{\sqrt{d_\sigma}\phi_{u_i,u_j}\}$ is an orthonormal basis for the unique $\sigma^* \otimes \sigma$-type $V_\sigma \otimes V_\sigma$ in $L^2(G)$. That is,

$$F(g) = \sum_\sigma\sum_{i,j} c_{\sigma,i,j}\sqrt{d_\sigma}\phi_{u_i,u_j}(g),$$

where

$$c_{\sigma,i,j} = \langle F, \sqrt{d_\sigma}\phi_{u_i,u_j}\rangle$$

and

$$\langle F, F\rangle = \sum_\sigma\sum_{i,j}|c_{\sigma,i,j}|^2.$$

If irreducible π and π' share a matrix coefficient, then π and π' are equivalent.

Sum of Squares Formula: $|G| = \sum_\sigma d_\sigma^2$, where σ ranges over a complete set of inequivalent, irreducible representations of G. We also have that d_σ divides $|G|$, but do not prove it in this course.

Schur Orthogonality Relations for Characters: Let (π, V) and (π', V') be irreducible, unitary representations of G. Then

$$\frac{1}{|G|}\sum_{g\in G}\chi_\pi(g)\overline{\chi_{\pi'}(g)} = \begin{cases}1 & \pi\cong\pi'\\ 0 & \pi,\pi' \text{ inequivalent}\end{cases}$$

Peter-Weyl Theorem for Characters (General Finite Case): We have two orthonormal bases for $Class(G)$, the class functions on G. First, we have the character basis

$$B = \{\chi_\sigma\},$$

where σ ranges over a complete set of inequivalent, irreducible representations of G. Then we have the basis

$$C = \{\sqrt{|G|/|C_x|}e_x\},$$

where e_x is the indicator function on the conjugacy class for x; that is, $e_x(g) = 1$ if g is in C_x, $= 0$ otherwise.

If F is in $Class(G)$, then

$$F(g) = \sum_\sigma c_\sigma \chi_\sigma(g),$$

where

$$c_\sigma = \langle F, \chi_\sigma \rangle$$

and

$$\langle F, F \rangle = \sum_\sigma |c_\sigma|^2.$$

Counting Formula for Irreducibles: The number of equivalence classes for irreducible representations of G equals the number of conjugacy classes in G.

Classification by Characters: Irreducible representations (π, V) and (π', V') are equivalent if and only if $\chi_\pi = \chi_{\pi'}$. Applying full reducibility to the representation (π, V), we have

$$V = V_{\sigma_1}^{n_1} \oplus \cdots V_{\sigma_k}^{n_k}$$

if and only if

$$\chi_\pi = \sum_{i=i}^{k} n_i \chi_{\sigma_i}.$$

Multiplicity Formulas: If $\chi_\pi = \sum_\sigma n_\sigma \chi_\sigma$, then

(1) $\langle \chi_\pi, \chi_\sigma \rangle = n_\sigma$, and
(2) $\langle \chi_\pi, \chi_\pi \rangle = \sum_\sigma n_\sigma^2$.

To find the projection operators onto each space of σ-types, we use convolution:

Projection Formula for Functions: The orthogonal projection to the matrix coefficient space of irreducible σ is defined by

$$P_\sigma f = d_\sigma f * \chi_\sigma = d_\sigma \chi_\sigma * f.$$

Projection Formula for Representations: Let V^σ be the subrepresentation of σ-types in V. The orthogonal projection $P_\sigma : L^2(G) \to V^\sigma$ is defined by

$$P_\sigma v = d_\sigma \pi(\overline{\chi_\sigma})v,$$

where

$$\pi(f)v = \frac{1}{|G|}\sum_{g\in G} f(g)\pi(g)v.$$

From this we obtain a Spectral Theorem analogue for representations:

$$\pi(g) = \sum_{\sigma} \sigma(g)P_{\sigma}.$$

Plancherel Formula: Let f be in $L^2(G)$. Then

$$\langle f, f\rangle = \sum_{\sigma} Trace(\sigma(f)\sigma(f)^*),$$

where σ ranges over a set of representatives for the irreducible classes.

Character Formula for Alternating and Symmetric 2-Tensors: Let (π, V) be a representation of G with character χ. Then $(\pi_{alt}, \wedge^2 V)$, the representation on alternating 2-tensors of V, has character

$$\chi_{alt}(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2)).$$

For $(\pi_{sym}, \odot^2 V)$, the representation on symmetric 2-tensors on V, the character is

$$\chi_{sym}(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2)).$$