INTRODUCTION TO GROUP REPRESENTATIONS
JUNE 11, 2012
PROBLEM SET 1

1. Let H be any group. Show that the following are group actions of G on X and find the corresponding orbits:

(a) $G = H$ acts on $X = H$ by $R(h)(x) = xh^{-1}$,

(b) $G = H$ acts on $X = H$ by $L(h)(x) = hx$,

(c) $G = H$ acts on $X = H$ by $D(h) = hxh^{-1}$, and

(d) $G = H \times H$ on $X = H$ by $J(h_1, h_2)(x) = h_1 x h_2^{-1}$.

2. (a) Consider the set of matrices $R(\theta) = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix}$ for θ in $[0, 2\pi)$. Show that these matrices commute and find a basis that simultaneously diagonalizes them over the complex numbers. (Hint: Euler's Formula)

(b) Repeat (a) with the set of matrices $H(t) = \begin{pmatrix} cosh(t) & sinh(t) \\ sinh(t) & cosh(t) \end{pmatrix}$ for all real t. Note that $cosh(t) = (e^t + e^{-t})/2$ and $sinh(t) = (e^t - e^{-t})/2$.

3. Let A be the complex matrix $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

(a) Find the smallest positive m such that $A^m = I$, and verify the Cayley-Hamilton Theorem for each A^k.

(b) Find the minimal polynomial for each power of A, and find a joint eigenbasis for $\mathbb{C}^4$ with respect to the set of all A^k.

4. (a) Verify that $GL(n, \mathbb{C}) = \{n \times n$ invertible matrices with complex entries$\}$ is a group under matrix multiplication.

(b) Verify that $SL(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C})$ such that $det(A) = 1\}$ is a normal subgroup of $GL(n, \mathbb{C})$.

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5. Let (π, V) be a representation of the finite group G. Verify that

(a) $\pi(e) = I$,

(b) $[\pi(g)]^{-1} = \pi(g^{-1})$ for all g in G, and

(c) $[\pi(g)]^k = \pi(g^k)$ for all $k \in \mathbb{Z}$.

6. Verify the homomorphism properties for π_1 and π_2 of S_3. Also verify Problem 5 for (23) and (123). Show that $\pi_2 = det(\pi_1)$. Recall that π_1 is the permutation action on $\mathbb{C}^3$ and π_2 is the sgn representation.

7. Verify that (π_1, V_2) has no one-dimensional subrepresentations (equivalently, no one-dimensional G-invariant subspaces in V_2). Recall that π_1 is the permutation action of S_3 on $\mathbb{C}^3$ and $V_2 = \{(a, b, c) : a + b + c = 0\}$.

(Hint: Suppose the span of (x, y, z) is invariant under S_3. Then $\pi_1(\sigma)(x, y, z) = \lambda_\sigma(x, y, z)$ for each σ of order 2. Show that $\lambda_\sigma = -1$ for some σ implies that $x = y = z = 0$.)

8. (a) If G is finite abelian, verify that $\pi(G)$ is abelian.

(b) Find all one-dimensional representations of $\mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4$, and $\mathbb{Z}/2 \times \mathbb{Z}/2$.

(c) Does $\mathbb{Z}/2 \times \mathbb{Z}/2$ have any faithful one-dimensional representations? Recall that faithful means the kernel of π is trivial.

9. Let $G = \mathbb{R}$ and $V = \mathbb{C}^2$. If $\pi(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, show that the only one-dimensional subspace invariant under G is the span of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

10. (a) Let N be a normal subgroup of G. Show that every representation of G/N extends to a representation of G using the quotient homomorphism $q : G \to G/N$.

(b) Use Part (a) to find three non-trivial one-dimensional representations of D_8, the symmetry group of the square.

(c) Use Part (a) to find three non-trivial one-dimensional representations of Q, the quaternionic group.

11. (a) If σ is an element of $Aut(G)$ and π a representation of G, show that $\pi \circ \sigma$ is also a representation of G of the same dimension.

(b) Show that one-dimensional representations are unchanged by the action in (a) under inner automorphisms.

(c) Show that $Aut(G)$ acts on the set of one-dimensional representations π of G by $\sigma(\pi) = \pi \circ \sigma^{-1}$. By (b), this action descends to $Out(G) = Aut(G)/Inn(G)$, the group of outer automorphisms.

(d) Find the orbits of the $Out(G)$ action on the sets of one-dimensional representations in Problem 8(b) and 10(b) and (c).

Notes:

1. Problems 1 and 11 are group action review. Problem 11 is possibly hard and not necessary, but part (c) contains the idea that a group action on X transfers to mappings on X by adding or removing the inverse.

2. Problem 2 reviews simultaneous diagonalization and Problem 3 reviews this, Cayley-Hamilton Theorem, and minimal polynomials.

Here are the general results of interest for us. They apply to general theory and are not so necessary for the exercises:

Theorem 1. *Let $T : V \to V$ be a linear transformation on a finite-dimensional complex vector space V. Then there exists a basis of eigenvectors for T (that is, T can be put in diagonal form) if and only if the minimal polynomial of T factors into distinct linear factors. The latter means that $m_T(x) = (x - c_1) \ldots (x - c_k)$ with no repeated roots.*

Since $\pi(g)^N = I$ for some N and $p(x) = x^N - 1$ has distinct roots over $\mathbb{C}$, this result shows that each $\pi(g)$ for a representation can be put into diagonal form in the context of finite groups; that is, the theory does not require Jordan or rational canonical forms from advanced linear algebra.

Theorem 2. *Let $T_\alpha : V \to V$ be a collection of linear transformation on a finite-dimensional vector space V indexed by α. Suppose each T_α is diagonalizable. Then there exists a joint eigenbasis for the set $\{T_\alpha\}$ if and only if $T_\alpha T_\beta = T_\beta T_\alpha$ for all $\alpha, \ \beta$. A joint eigenbasis is an eigenvector basis that puts all T_α into diagonal form at the same time.*

This result show that representations of finite abelian groups are relatively uncomplicated. There always exists a single basis such that all $\pi(g)$ are represented by diagonal matrices; thus they can be understood entirely through their one-dimensional representations.

3. Aside from 7, the remainder are meant to be straightforward applications.

4. For 10(b), label the upper right hand corner of a square as 1, and continue counterclockwise. In cycle notation,

$$D_8 = \{e, (1234), (13)(24), (1432), (13), (24), (12)(34), (14)(23)\}.$$

If we let $c = (14)(23)$ and $r = (1234)$, then every element of D_8 is of the form $c^i r^j$ with i in $\{0, 1\}$ and j in $\{0, 1, 2, 3\}$. D_8 is then completely described by the generators and relations

$$c^2 = e, \ r^4 = e, \ crc = r^{-1}.$$

If we have generators and relations for a group, checking the homomorphism property requires less work. For D_8, we have a homomorphism if we verify

$$[\pi(c)]^2 = e,\ [\pi(r)]^4 = e,\ \text{and}\ \pi(c)\pi(r)\pi(c) = [\pi(r)]^{-1}.$$

This applies to Problem 6 as well; if $c = (23)$ and $r = (123)$, then S_3 is completely described by

$$c^2 = e,\ r^3 = e,\ crc = r^{-1}.$$

5. For 10(c), we recall the definition of the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$. Consider the list $ijki$. If we multiply any two adjacent letters to the right, we get the next letter; going to the left, we get the next letter with a minus sign. If we square any symbol, we get -1. That is,

$$i^2 = j^2 = k^2 = -1,\ ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik.$$

Every subgroup of Q is normal, and we are interested in the three subgroups $N_i = \{\pm 1, \pm i\}$, N_j, and N_k.

6. Next problem set will have a review of complex linear algebra, unitary matrices, and the Spectral Theorem.