INTRO TO REP THEORY - JUNE 18, 2012
PROBLEM SET 2
RT2. UNITARY REPRESENTATIONS

1. Define the standard Hermitian inner product on $\mathbb{C}^n$ as follows: if $v = (x_1, \ldots, x_n)$ and w $= (y_1, \ldots y_n)$ are column vectors, then

$$\langle v, w\rangle = w^* v = \sum x_i \overline{y_i},$$

where $\overline{z}$ is complex conjugation and w^* is the conjugate transpose of w; this is the row vector $w^* = (\overline{y_1}, \ldots, \overline{y_n})$.

(a) Verify the following properties of $\langle \cdot, \cdot \rangle$: suppose c in $\mathbb{C}$ and v_1, v_2, w in $\mathbb{C}^n$.

(1) $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$,
(2) $c\langle v, w\rangle = \langle cv, w\rangle = \langle v, \bar{c}w\rangle$,
(3) $\langle v, w\rangle = \overline{\langle w, v\rangle}$,
(4) $||v||^2 = \langle v, v\rangle \geq 0$ and equals 0 iff $v = 0$, and
(5) $||cv|| = |c|||v||$.

(b) If L is an $n \times n$ matrix over $\mathbb{C}$, show that the adjoint of L with respect to $\langle \cdot, \cdot \rangle$ equals the conjugate transpose L^*. That is, show that $\langle Lv, w\rangle = \langle v, L^* w\rangle$ for all v, w in $\mathbb{C}^n$.

(c) For matrices L_1, L_2 as in (b), show that $(L_1 L_2)^* = L_2^* L_1^*$.

2. (a) Show that $v_1 = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$ form an orthonormal basis of $\mathbb{C}^2$ with respect to $\langle \cdot, \cdot \rangle$.

(b) Verify Fourier's Trick and Parseval's Identity for $v = \begin{pmatrix} i \\ i \end{pmatrix}$. That is, if $v = c_1 v_1 + c_2 v_2$, then $c_i = \langle v, v_i\rangle$ and $||v||^2 = |c_1|^2 + |c_2|^2$.

3. Decompose $(2, i)$ as sum of vectors parallel and orthogonal to $(1 + i, -i)$.

4. (a) Show that the following statements are equivalent:

(1) U is a unitary matrix: $UU^* = U^*U = I$, where U^* is the conjugate transpose of U. That is, $[u_{ij}]^* = [\overline{u_{ji}}]$.
(2) $\langle Uv, Uw\rangle = \langle v, w\rangle$ for all v, w in $\mathbb{C}^n$,
(3) the columns of U form an orthonormal basis of $\mathbb{C}^n$, and
(4) the rows of U form an orthonormal basis of $\mathbb{C}^n$.

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(b) Show that the set of $n \times n$ unitary matrices $U(n) = \{U \in GL(n, \mathbb{C}) \mid UU^* = I\}$ form a group, and that the set $O(n)$ of real orthogonal matrices is a subgroup of $U(n)$.

(c) Show that $SU(n) = \{U \in U(n) \mid det(U) = 1\}$ is a normal subgroup of $U(n)$.

5. (Spectral Theorem for Hermitian matrices) A square matrix X is called Hermitian iff $X^* = X$.

(a) Show that eigenvalues for Hermitian matrices are real.

(b) Show that eigenvectors for distinct eigenvalues of a Hermitian matrix are orthogonal.

(c) Show that if W is a subspace invariant under Hermitian X then the orthogonal complement $W^\perp$ is also X-invariant. Recall that a subspace W is called invariant under X if Xw is in W for all w in W.

Also recall that $W^\perp$, the orthogonal complement of W, is the subspace of all v such that $\langle v, w \rangle = 0$ for all w in W.

(d) Show that Hermitian X can be put into diagonal form using an orthonormal basis. Thus there exists a unitary matrix U such that $U^*XU = D$ where D is diagonal with real entries.

(e) Diagonalize the Hermitian matrix $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ using an orthonormal basis.

6. (a) Show that eigenvalues for unitary matrices satisfy $|\lambda| = 1$.

(b) Show that eigenvectors for distinct eigenvalues of a unitary matrix are orthogonal.

(c) Show that if W is a subspace invariant under an invertible matrix A then W is invariant under A^{-1}. Use this to show that if W is invariant under unitary U then $W^\perp$ is also invariant under U.

(d) Show that unitary U can be put into diagonal form using an orthonormal basis. Thus there exists a unitary matrix W such that $W^*UW = D$ where D is diagonal with diagonal entries on the unit circle in $\mathbb{C}$.

(e) Diagonalize the unitary matrix $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ using an orthonormal basis.

7. Let $A = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}$. Diagonalize A using an orthonormal basis.

Find the orthogonal projections P_λ for each eigenspace and verify

(1) $P_\lambda^2 = P_\lambda$,
(2) $P_{\lambda_1} P_{\lambda_2} = P_{\lambda_2} P_{\lambda_1} = 0$,
(3) $P_{\lambda_1} + P_{\lambda_2} = I$, and
(4) $A = \lambda_1 P_{\lambda_1} + \lambda_2 P_{\lambda_2}$ and A commutes with each P_λ.

Apply each P_λ to $e_1 = (1, 0)$. Verify that $e_1 = P_{\lambda_1} e_1 + P_{\lambda_2} e_1$ and that

$$Ae_1 = \lambda_1 P_{\lambda_1} e_1 + \lambda_2 P_{\lambda_2} e_2.$$

(Hint: Verfiy that $v_1 = (\sqrt{2}/2, \sqrt{2}/2)$ and $v_2 = (\sqrt{2}/2, -\sqrt{2}/2)$ are unit eigenvectors for A.)

8. Let A be an invertible $n \times n$ matrix with complex entries. Define $A^0 = I$.

(a) Define $\pi : \mathbb{Z} \to \mathbb{C}^n$ by $\pi(k)v = A^k v$ for all v in $\mathbb{C}^n$. Show that π is a representation of $\mathbb{Z}$.

(b) What can we say if A is

(1) diagonalizable,
(2) Hermitian, or
(3) unitary?

What if the kernel of π is non-trivial?

9. For the group D_8, center the square at the origin of $\mathbb{R}^2$. Then the symmetries are real linear transformations of $\mathbb{R}^2$. The matrices with respect to the standard basis vectors also act on $\mathbb{C}^2$.

(a) Show that this two-dimensional representation is unitary with respect to the standard Hermitian inner product on $\mathbb{C}^2$ and irreducible.

(b) Find orthonormal bases of eigenvectors for $\pi(c)$ and $\pi(r)$.

10. (a) Repeat 9 with an equilateral triangle centered at the origin, vertex 1 on the positive real axis, labels continuing counter-clockwise. Replace D_8 with S_3.

(b) Find orthonormal bases of eigenvectors for $\pi(c)$ and $\pi(r)$.

11. Let $X = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Consider the representation

$$\pi(g) = \begin{pmatrix} \chi_2(g) & 0 \\ 0 & \chi_3(g) \end{pmatrix}$$

of $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ from Problem 8(b) of Solution Set 1.

(a) Compute

$$L = \frac{1}{|G|} \sum_{g \in G} \pi(g)^* X \pi(g).$$

If $\langle \cdot, \cdot \rangle$ is a the standard Hermitian inner product on $\mathbb{C}^2$, show that

$$\langle v, w \rangle_1 = \langle Xv, w \rangle$$

is not invariant under G, but

$$\langle v, w \rangle_2 = \langle Lv, w \rangle$$

is invariant under G.

(b) Repeat (a) for $G = \mathbb{Z}/4$.

Notes:

1. Problems 1-6 are complex analogues of results seen at the end of a first year Linear Algebra course, typically finishing with the Spectral Theorem (diagonalizing a real symmetric matrix with an orthonormal basis). The results are essentially the same if we replace "symmetric" with "Hermitian" and "orthogonal" with "unitary." We could add a Gram-Schmidt process problem (orthonormailzing a basis) to Problem 3, but this set is already too long and I'm pretty sure we don't use it.

One drawback to the real theory is that orthogonal matrices are not always diagonalizable; consider the rotation by $\pi/2$ counter-clockwise about the origin of $\mathbb{R}^2$. When working over the complex numbers, unitary matrices are diagonalizable, which makes our work somewhat smoother.

Problems 5 and 6 may be a bit much, so it may be better to wait until the solutions. Note that 5(b) and 6(b) are false leads for the proofs of the main results in (d); while orthogonality of eigenspaces is important, we only get a proof if each eigenspace is one-dimensional.

2. Problem 7 is the Spectral Theorem at full power. The projections will be diagonalizable with eigenvalues 1 and 0; if v_1 is a unit eigenvector to project to and v_2 a unit eigenvector for the other eigenvalue, set $P = [v_1 v_2]$ and

$$P_\lambda = P \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} P^{-1}.$$

2. Problem 8 notes that the business of working with polynomials of matrices is really representation theory of the integers; we need ring theory to exploit this idea properly, and this subject is better left to advanced linear algebra.

3. For Problems 9 and 10, the work is simplified using generators and relations for each group as before.

4. Problem 11 checks that the averaging technique does produce an invariant inner product.