# INTRO TO GROUP REPS - JULY 2, 2012
# PROBLEM SET 4
# RT4. CONSTRUCTIONS FROM LINEAR ALGEBRA

1. Suppose $(\pi, V)$, $(\pi', V')$ and $(\pi'', V'')$ are representations of $G$. Suppose $L_1 : V \to V'$ and $L_2 : V' \to V''$ are equivalences.

(a) Show that $L_1^{-1} : V' \to V$ is an equivalence.

(b) Show that $L_2 \circ L_1 : V \to V''$ is an equivalence.

2. Define an equivalence relation on representations of $G$ by $(\pi, V) \sim (\pi', V')$ if and only if there exists an equivalence $L : V \to V'$.

(a) Show that $\sim$ is an equivalence relation on representations.

(b) Repeat (a) for unitary equivalences and unitary representations.

(c) Show that $Out(G)$ acts on equivalence classes of representations of $G$ by the action $\sigma(\pi) = \pi \circ \sigma^{-1}$.

3. Let $(\pi, V)$ and $(\pi', V')$ be representations of $G$. Verify the following group actions of $G$ (note that we should also verify linearity for each $\pi(g)$):

(a) Direct sum $V \oplus V' : \pi_{\oplus}(g)(v, v') = (\pi(g)v, \pi'(g)v')$,

(b) Dual space $V^* : \pi^*(g)v^*(w) = v^*(\pi(g^{-1})w)$,

(c) Linear transformations $Hom_{\mathbb{C}}(V, V') : [\sigma(g)L](v) = \pi'(g)[L(\pi(g^{-1})v)]$, and

(d) Tensors $V \otimes V'$ : on monomials, $(\pi \otimes \pi')(g)(v \otimes v') = \pi(g)v \otimes \pi'(g)v'$.

4. Suppose the $\pi$ and $\pi'$ in Problem 3 are also unitary. Verify that the following are also unitary; that is, check that the induced forms are inner products and that they are invariant under the group action:

(a) Direct sum $V \oplus V' : \langle (v, v'), (w, w') \rangle = \langle v, w \rangle + \langle v', w' \rangle'$,

(b) Dual space $V^* : \langle \langle \cdot, v \rangle, \langle \cdot, w \rangle \rangle_* = \langle w, v \rangle$,

(c) Linear transformations $Hom_{\mathbb{C}}(V, V') : \langle L_1, L_2 \rangle = Tr(L_1 L_2^*)$,

(d) Tensors $V \otimes V'$ : on monomials, $\langle v \otimes v', w \otimes w' \rangle = \langle v, w \rangle \langle v', w' \rangle'$.

5. (a) Let $(\pi, V)$ be a representation of $G$. If $f_1, f_2$ are in $V^*$ and $c_1$ in $\mathbb{C}$, show that $c_1 f_1 + f_2$ is also in $V^*$.

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(b) Let $B^* = \{e_1^*, e_2^*, e_3^*\}$ be the standard basis for $(\mathbb{C}^3)^*$. That is, $e_i^*(e_j) = 1$ if $i = j$ and 0 otherwise. Using the permutation representation of $S_3$ on $\mathbb{C}^3$, compute $\pi^*(g)e_1^*$ for all $g$ is $S_3$.

(c) In part (b), if $L(x, y, z) = 2x + y - 2z$, find a vector such that $L(v) = \langle v, w\rangle$ for all $v$ in $\mathbb{C}^3$. Compute $\pi^*(23)L$ and $\pi^*(123)L$, and verify that the length of $L$ is unchanged.

6. Suppose $(\pi, V)$ is unitary and $(\pi, W)$ is a subrepresentation.

(a) Show that the orthogonal projection $P_W : V \to W$ is an intertwining operator,

(b) Show that $V/W^{\perp}$ is equivalent to $W$ as representations.

(c) If $(\pi, \mathbb{C}^3)$ is the permutation representation of $S_3$ and $W$ is the span of $(1, 1, 1)$. Find $P_W$ and $P_{W^{\perp}}$ explicitly, and verify the intertwining property.

7. Suppose $(\pi, V)$ and $(\pi', W)$ are representations of $G$.

(a) Verify that $L : V^* \otimes W \to Hom_{\mathbb{C}}(V, W)$ defined by extending

$$L(v^* \otimes w)(v') = v^*(v')w$$

is an equivalence of representations of $G$.

(b) If the representations in (a) are unitary, verify that $L$ is a unitary equivalence.

(c) Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be the linear transformation defined by

$$T(x, y, z) = (2x + z, x + y, x - y - z).$$

Compute $\sigma(g)T$ for $g = (23), (123)$ in $S_3$ using the permutation action on $\mathbb{C}^3$.

8. Let $(\pi, W)$ be the two-dimesional, irreducible, unitary representation of $S_3$, where

$$W = \{(x, y, z) \mid x + y + z = 0\}$$

in $\mathbb{C}^3$ with the permutation action by $S_3$. Consider the induced unitary representation on $W \otimes W$ . Let

$$v_1 = (1, -1, 0), \quad v_2 = (1, 0, -1), \quad v_3 = (0, 1, -1).$$

We decompose $W \otimes W$ as an orthogonal direct sum of irreducibles

$$W = W_1 \oplus W_2 \oplus W_3.$$

Show that

(a) $W_1 = \mathbb{C}(v_1 \otimes v_2 - v_2 \otimes v_1)$ is a subrepresentation and is equivalent to the sgn representation of $S_3$,

(b) $W_2 = \mathbb{C}(v_1 \otimes v_1 + v_2 \otimes v_2 + v_3 \otimes v_3)$ is a subrepresentation and is equivalent to the trivial representation of $S_3$,

(c) $W_3 = Span_{\mathbb{C}}(v_1 \otimes v_1 - v_2 \otimes v_2, v_2 \otimes v_2 - v_3 \otimes v_3)$ is a subrepresentation and equivalent to the irreducible two-dimensional representation of $S_3$,

(d) the subspaces in (a)-(c) are mutually orthogonal with respect to the inner product on tensors, and

(e) let $B$ be the basis of $W \otimes W$ using the vectors from (a)-(c). Find $[\pi \otimes \pi(g)]_B$ for $g = (23), (123)$. (To get a unitary matrix, we need to use an orthonormal basis.)

9. In Problem 8, express each tensor in terms of the standard basis $B = \{e_i \otimes e_j\}$ for $\mathbb{C}^3 \otimes \mathbb{C}^3$, and recheck the subrepresentation and orthogonal properties.

10. Let $(\pi, \mathbb{C}^2)$ be the irreducible two-dimensional representation of $D_8$. Show that $\chi_i \otimes \pi$ is equivalent to $\pi$ for each character $\chi_i$ of $D_8$. Also show that $\pi$ is equivalent to $\pi^*$.