INTRO TO GROUP REPS - JULY 16, 2012
PROBLEM SET 6
RT6. FUNCTION SPACES

1. Suppose σ be a group action of G on a finite set X. Let V be a vector space with basis $B = \{e_x \mid x \in X\}$. We define a representation (π, V) by

$$\pi(g) \sum_{x \in X} c_x e_x = \sum c_x e_{\sigma(g)x}.$$

(a) Show that $\pi(g)$ is a permutation matrix with respect to the basis B. That is, $\pi(g)$ has all zero entries save for one 1 in each row and column.

(b) Show that (π, V) is unitary with respect to the Hermitian inner product

$$\langle v, w \rangle = \sum_{x \in X} c_x \overline{d_x}$$

if $v = \sum c_x e_x$ and $w = \sum d_x e_x$.

(c) Show that the trivial representation occurs in V with multiplicity equal to the number of orbits in X.

(d) Show that (π^*, V^*) is equivalent to $(L, L^2(X))$, where $L^2(X)$ is the inner product space of functions on X.

(e) Show that $\chi_\pi(g)$ is equal to the number of x in X that are fixed points for $\sigma(g)$. That is, $\chi_\pi(g)$ is the number of x in X such that $\sigma(g)x = x$. See Problem 8 for the definition of χ_π.

2. Let $G = X = \{\pm 1\}$ and let $L^2(G)$ be the vector space of functions from G to $\mathbb{C}$. Let G act on $L^2(G)$ by left translation: $[L(g)f](x) = f(g^{-1}x)$, and we define an invariant Hermitian inner product on $L^2(G)$ by

$$\langle f, h \rangle = \frac{1}{2}[f(1)\overline{h(1)} + f(-1)\overline{h(-1)}].$$

(a) Verify that $(L, L^2(G))$ is a unitary representation.

(b) Verify that $B = \{\chi_{triv}, \chi_{sgn}\}$ is an orthonormal basis for $L^2(G)$. The χ's are the characters of G from SS1, Problem 8(a).

(c) Show that

$$[P_{triv} f](x) = \frac{1}{2}(f(x) + f(-x)) = \langle f, \chi_{triv} \rangle \chi_{triv}(x)$$

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and

$$[P_{sgn}f](x) = \frac{1}{2}(f(x) - f(-x)) = \langle f, \chi_{sgn}\rangle \chi_{sgn}(x)$$

are orthogonal projections.

(d) Let $f(1) = 3,\ f(-1) = -4$. Decompose using (c), and also apply Fourier's Trick and Parseval's Identity to f with respect to the basis in (b).

3. Let $X = \{1, 2, 3, 4\}$ and consider the permutation representation π of D_8 on $L^2(X) \cong (\mathbb{C}^4)^*$; that is, realize D_8 as a subgroup of S_4 using $r = (1234)$ and $c = (14)(23)$. Decompose $L^2(X)$ into irreducible subrepresentations, and verify that they are orthogonal with respect to the invariant inner product on $L^2(X)$. (Hint: SS1, Problems 3(a) and 10(b))

4. Suppose $F : \mathbb{R} \to \mathbb{C}$ is piecewise continuous; that is, $F(t) = f(t) + ig(t)$ with $f, g : \mathbb{R} \to \mathbb{R}$ piecewise continuous. We define $\int F(t)dt = \int f(t)dt + i\int g(t)dt$ if all terms make sense.

We define $C_p(S^1)$ as the space of piecewise continuous functions $F : \mathbb{R} \to \mathbb{C}$ that are periodic of period 2π; that is,

$$F(t + 2\pi) = F(t).$$

Here $G = S^1$, the circle group, represented here as the quotient group $\mathbb{R}/2\pi\mathbb{Z}$. We let t' in S^1 act on $C_p(S^1)$ by

$$[L(t')F](t) = F(t - t').$$

Furthermore, we define a Hermitian inner product on $C_p(S^1)$ by

$$\langle f, h\rangle = \int_0^{2\pi} f(t)\overline{h(t)}dt.$$

(a) Show that $\langle \cdot, \cdot\rangle$ is a Hermitian inner product, and that $(L, C_p(S^1))$ is a unitary representation of S^1. (Ignore any analytic issues.)

(b) Show that the span of $F_n(t) = e^{int}$ is a subrepresentation of L of type χ_{-n} (resp. of R of type χ_n).

(c) Show that $B = \{F_n\}$ forms an orthonormal set, where $n \in \mathbb{Z}$. (In fact, B is a Hilbert basis for $L^2(S^1)$, but we assume no analysis.)

5. (a) Assuming B is an orthonormal "basis" in Problem 3, apply Fourier's Trick to the square wave function (extended periodically)

$$F(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases};$$

that is, represent $F(t)$ by a Fourier series $\sum c_n F_n(t)$ with $c_n = \langle F, F_n\rangle$.

(b) Apply Parseval's Identity to part (a).

(c) Use part (b) to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Verify by summing the first 10, 100, 500, 750 terms.

6. We define the trace of a square matrix A as the sum of its diagonal entries. With respect to the standard inner product on $\mathbb{C}^n$,

$$Trace(A) = \sum_i \langle Ae_i, e_i \rangle.$$

In fact, this formula holds with any orthonormal basis. In general, if $B = \{u_i\}$ is a basis for V and $B^* = \{u_i^*\}$ is the corresponding dual basis, then the trace of a linear transformation $T : V \to V$ is given as

$$Trace(T) = \sum_i u_i^*(Tu_i).$$

(a) Using induction, prove that $Trace(AB) = Trace(BA)$ with A and B square matrices.

(b) Find a counterexample to $Trace(ABC) = Trace(ACB)$. Note that, by (a),

$$Trace(ABC) = Trace((BC)A) = Trace(C(AB)).$$

7. (a) Show that $Trace(PAP^{-1}) = Trace(A)$. Use this to show that $Trace(T)$ is well-defined; that is, $Trace(T)$ is independent of the basis chosen.

(b) If A is diagonalizable, show that $Trace(A)$ is the sum of the eigenvalues of A, counting multiplicities.

(c) If $p_A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is the characteristic polynomial of A, show that $Trace(A) = -a_{n-1}$.

(d) Not assuming diagonalizability, use (c) to show that $Trace(A)$ is the sum of the eigenvalues of A, counting multiplicities.

8. Let (π, V) and (π', V') be representations of G. The character of π, denoted $\chi_\pi : G \to \mathbb{C}$, is defined by $\chi_\pi(g) = Trace(\pi(g))$.

(a) Show that $\chi_\pi(e)$ is the dimension of V.

(b) If π and π' are equivalent, show that $\chi_\pi = \chi_{\pi'}$ are equal. (The converse is true; we prove it later.) Also show that χ_π is constant on conjugacy classes of G.

(c) Show that $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$.

(d) Show that

(1) $\chi_{\pi \oplus \pi'} = \chi_\pi + \chi_{\pi'}$,
(2) $\chi_{\pi^*} = \overline{\chi_\pi}$, and

(3) $\chi_{\pi\otimes\pi'} = \chi_\pi \cdot \chi_{\pi'}$.

9. Let $G = A_4$, the alternating group on 4 letters, and consider the representation $(\pi, \mathbb{C}^3)$ induced by rigid motions of a tetrahedron centered at the origin of $\mathbb{R}^3$.

(a) Find all conjugacy classes of A_4. We have three characters. How many irreducible classes remain, and what is their dimension?

(b) Find $\chi_\pi(g)$ for each g by finding the eigenvalues of each symmetry by observation and sum. Show that χ_π is orthogonal to each character in $L^2(A_4)$.

(c) To describe π on $\mathbb{R}^3$, we first identify the tetrahedron in a cube with vertices at $(\pm1, \pm1, \pm1, \pm1)$. Let

$$v_1 = (1,1,1), \quad v_2 = (-1,-1,1), \quad v_3 = (1,-1,-1), \quad v_4 = (-1,1,-1)$$

be the vertices of a tetrahedron centered at the origin. If B is the standard basis of $\mathbb{R}^3$, find $[\pi(g)]_B$ for $g = (12)(34),\ (123)$ and verify part (b).

(d) Show that the representation in (c) is irreducible.

(e) Set up the (trace) character table of A_4 and verify orthonormality of the rows.

10. Let c be the group action of G on G by conjugation; that is, $c(g)(x) = gxg^{-1}$. Let $Class(G)$ be the subspace of $L^2(G)$ on which conjugation acts trivially. That is, f is in $Class(G)$ if and only if

$$f(gxg^{-1}) = f(x) \quad \text{for all} \quad g, x \in G.$$

(a) If f is in $Class(G)$, show that $f(gx) = f(xg)$.

(b) If f is in $L^2(G)$, show that

$$(Af)(x) = \frac{1}{|G|} \sum_g f(gxg^{-1})$$

is in $Class(G)$.

(c) Find an orthonormal basis for $Class(G)$ in terms of the conjugacy classes of G. See Problem 1(c).

(d) If (π, V) is a representation of G, show that χ_π is in $Class(G)$.

(e) Compare (c) and (d) for $G = S_3$, D_8, Q, and A_4. What if G is abelian? Conjecture for non-abelian?