INTRO TO GROUP REPS - JULY 23, 2012
PROBLEM SET 7
RT7. FINITE ABELIAN GROUPS

1. Suppose G is a finite abelian group. We show that the columns of the character table for G must form an orthonormal basis of $\mathbb{C}^n$ in two ways.

(a) Show column orthogonality by considering orthogonality of the rows.

(b) Fix g in G. Define $e_g : G^* \to \mathbb{C}$ by $e_g(\chi) = \chi(g)$. Show that e_g is an element of $G^{**} = (G^*)^*$, the character group of the character group, and that $g \mapsto e_g$ defines an isomorphism from G to G^{**}.

(c) Explain why (b) shows column orthogonality.

2. Let G be a finite abelian group such that each χ in G^* satisfies $\chi = \overline{\chi}$.

(a) Describe G.

(b) What if G is non-abelian? In this case, G^* consists of the one-dimensional representations.

3. We call ω in $\mathbb{C}$ an n-th root of unity if $\omega^n = 1$. In other words, ω is a root of $p(z) = z^n - 1$. ω is called an primitive n-th root of unity if $\omega^k = 1$ for positive $k \leq n$, then $k = n$.

(a) Show that $\omega = e^{2\pi i/n}$ is a primitive n-th root of unity.

(b) If ω is an n-th root of unity, show that ω^k is also an n-th root of unity, and that there are n distinct n-th roots of unity. Describe the primitive n-th roots.

(c) Show that $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$ for all n-th roots of unity not equal to 1.

(d) Describe the faithful irreducible representations of $\mathbb{Z}/n$.

4. (a) Set up the character table for $\mathbb{Z}/n$.

(b) Verify directly that the rows and columns form orthonormal bases. (Note: complex conjugate the second vector!)

(c) Use Fourier's Trick to express

$$f_1(k) = \begin{cases} 1 & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

in the character basis for $G = \mathbb{Z}/n$ for $n = 2, 3, 4$. Verify Parseval's Identity in each case.

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5. Let $G = \mathbb{Z}/2 \times \ldots \mathbb{Z}/2$ (n factors).

(a) Describe all 2^n characters of G by noting that the standard basis $\{e_i\}$ over $\mathbb{Z}/2$ is a generating set for G.

(b) Use induction to verify orthogonality of characters directly.

(c) Describe each

$$\delta_g(h) = \begin{cases} 1 & h = g \\ 0 & \text{otherwise} \end{cases}$$

in terms of the character basis for $n = 2$ using Fourier's Trick, and verify Parseval's Identity.

6. Let G be a finite abelian group. Fix g in G, and consider the evaluation map $e_g : L^2(G) \to \mathbb{C}$ defined by $e_g(f) = f(g)$.

(a) Verify that e_g is a linear functional on $L^2(G)$, and show that $\{e_g\}$ forms a basis of the dual vector space $(L^2(G))^*$.

(b) With respect to the usual inner product on $L^2(G)$, verify that the function

$$\delta_g(h) = \begin{cases} 1 & h = g \\ 0 & \text{otherwise} \end{cases}$$

satisfies

$$f(g) = \langle f, |G|\delta_g \rangle$$

for all f in $L^2(G)$.

(c) Verify that $B = \{\sqrt{|G|}\delta_g\}$ is an orthonormal basis of $L^2(G)$, and that each $[L(g)]_B$ is a permutation matrix.

(d) For $G = \mathbb{Z}/2$, express each basis vector in B in terms of the character basis, and verify that the change of basis matrix is unitary.

7. Recall that G^*, the set of characters of G, is a group under multiplication of functions. Define a representation $(\pi, L^2(G))$ of G^* by

$$[\pi(\chi)f](g) = \chi(g)f(g).$$

(a) Verify that π is a unitary representation of G^* with respect to the usual inner product on $L^2(G)$.

(b) Show that each subspace $\mathbb{C}\delta_g$ in $L^2(G)$ is a subrepresentation of π of type e_g for G^*.

8. Derive the (trace) character tables for S_3, D_8, Q, and A_4 using only the one-dimensional representations. Recall that each column represents a conjugacy class, labeled by an element in the class and the number of elements in the class.

9. Let G be D_{12}, the dihedral group with 12 elements; that is, D_{12} is the symmetry group of a regular hexagon. Recall that D_{12} is described by generators and relations

$$r^6 = e, \ c^2 = e, \ crc = r^{-1}.$$

(a) Find all conjugacy classes, the center, and the commutator subgroup of G. Find all normal subgroups of G.

(b) Find all characters (one-dimensional representations) of G.

(c) Find two inequivalent, irreducible representations for G of dimension 2. Note that

$$12 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2 + 2^2.$$

(d) Set up the (trace) character table of G.

10. Fix odd $n \geq 5$. Let G be D_{2n}, the dihedral group with $2n$ elements. Repeat Problem 9 for G.