INTRO TO GROUP REPS - JULY 30, 2012
PROBLEM SET 8
RT7. FINITE ABELIAN GROUPS 2

For Problems 1-7, we assume G is finite abelian.

1. Let (π, V) a representation of G.

(a) Let χ be a one-dimensional representation of G. Using the $L^2(G)$ norm, show that $n_\pi = \langle \chi, \chi_\pi \rangle$ is the multiplicity of χ in π. Recall that $\chi_\pi(g)$ is the trace of $\pi(g)$.

(b) Verify part (a) for $(L, L^2(G))$.

2. Verify Problem 1(a) for the following restrictions to abelian subgroups.

(a) the irreducible two-dimensional representation of D_8 to $H_1 = \{e, r^2\}$, $H_2 = \{e, c\}$, and $H_3 = \langle r \rangle$.

(b) the irreducible two-dimensional representation of Q to $H_1 = \{\pm 1\}$ and $H_2 = \{\pm 1, \pm i\}$.

(c) the irreducible three-dimensional representation of A_4 to $H_1 = \{e, (12)(34), (13)(24), (14)(23)\}$ and $H_2 = \{e, (123), (132)\}$.

3. (a) For g, g' in G and f in $L^2(G)$, show that

(1) $\delta_g \cdot \delta_g = \delta_g$,
(2) $\delta_g \cdot \delta_{g'} = 0$ if $g \neq g'$,
(3) $f \cdot \delta_g = f(g) \cdot \delta_g$, and
(4) $f = \sum_g f(g)\delta_g$.

(b) If g, g' are in G, show that

$$L(g)\delta_{g'} = \delta_{gg'} = R({g'}^{-1})\delta_g,$$

and that

$$\delta_g * \delta_{g'} = \frac{1}{|G|}\delta_{gg'}.$$

(c) Verify that $f_1 * f_2 = f_2 * f_1$ and $(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$ for f_1, f_2, f_3 in $L^2(G)$.

(d) Verify part (b) when $G = \mathbb{Z}/2$ and $f_1 = \delta_0 + 2\delta_1$, $f_2 = 2\delta_0 + \delta_1$, and $f_3 = 2\delta_0$.

4. (a) Prove that $\widehat{(cf_1 + f_2)} = c\widehat{f_1} + \widehat{f_1}$.

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(b) Show that $\langle \widehat{f_1}, \widehat{f_1} \rangle_{L^2(B)} = \langle f_1, f_1 \rangle$.

(c) Prove the Convolution Formula: $\widehat{f_1 * f_2} = \widehat{f_1} \cdot \widehat{f_2}$.

(d) Verify (c) when f_1 and f_2 are characters.

(e) Verify (c) for f_1 and f_2 in Problem 3(c).

5. Let $f^*(g) = \overline{f(g^{-1})}$ for f in $L^2(G)$.

(a) Let $G = \mathbb{Z}/4$. Using the Projection Operator Formula for $L^2(G)$

$$P_\chi f = f * \chi = \chi * f,$$

compute P_{χ_1} with respect to the basis $\{\sqrt{|G|}\delta_g\}$ for $L^2(G)$.

(b) Verify that $\delta_1 = \sum_\chi P_\chi \delta_1$.

(c) Verify the Plancherel Formula:

$$\langle f, f \rangle = \sum_\chi \chi(f)\overline{\chi(f)}.$$

(d) Verify that $(f * f^*)(1) = \langle f, f \rangle$.

6. Suppose (π, V) is a unitary representation of G and f is in $L^2(G)$. Recall that $\pi(f) : V \to V$ is defined by

$$\pi(f)v = \frac{1}{|G|} \sum_{g \in G} f(g)\pi(g)v$$

(a) Verify that $\pi(L(g)f) = \pi(g)\pi(f)$ and $\pi(R(g)f) = \pi(f)\pi(g^{-1})$.

(b) Verify that $\pi(f * h) = \pi(f)\pi(h)$.

If $B = \{\chi\}$ is a character basis for $L^2(G)$, $f = \sum a_\chi \chi$, and $h = \sum b_\chi \chi$, show that

$$[\pi(f * h)]_B = [\pi(f)]_B [\pi(h)]_B$$

corresponds to multiplication of diagonal matrices.

(c) Show that $\pi(f^*) = \pi(f)^*$ with respect to the invariant inner product.

7. (a) Apply the Projection Operator Formula for representations

$$P_\chi v = \pi(\overline{\chi})v$$

to the representations of $\mathbb{Z}/4 = \langle r \rangle$ in Problem 2(a).

(b) Verify that the P_χ in (a) are orthogonal projections.

(c) Use P_χ to find each subrepresentation in (a). Interpret.

8. Let V be an inner product space. If $\langle v, v\rangle$ is known for all v in V, find a formula for $\langle v, w\rangle$ for v, w in V.

(Hint: First consider the real case. For the complex case, find the real part of $\langle v, w\rangle$ first, and then use $\langle iv, w\rangle$ to find the imaginary part.)

9. Let G be the nontrivial semidirect product of $\mathbb{Z}/4$ on $\mathbb{Z}/3$. That is, G is defined by generators and relations

$$y^4 = e,\ x^3 = e,\ yxy^{-1} = x^{-1}.$$

Note that all elements can be written in the form $y^i x^j$ $(0 \le i \le 3,\ 0 \le j \le 2)$. (For background, see Group Theory course, SS11, Problem 7(a).)

(a) Find the conjugacy classes, center, and commutator subgroup of G.

(b) Set up the character table of G, and verify the orthogonality of the rows and columns.

10. Fix $n = 2l$, and suppose $G = D_{2n}$ is the dihedral group with $2n$ elements. Repeat Problem Set 7, Problem 10.