INTRO TO GROUP REPS - AUGUST 6, 2012
PROBLEM SET 9
RT8. ANALYSIS ON FINITE GROUPS

1. (a) Prove that the columns of the character table for G are orthogonal (unweighted). What are the lengths squared of the columns?

(b) Verify for S_3, D_8, Q, A_4, and D_{12}.

2. (a) Let M be the matrix given by the character table of G. Compute $|det(M)|$.

(b) Verify for the groups in S_3 and A_4. What if G is abelian?

3. Here's the traditional proof of the Schur Orthogonality Relations. Suppose (π, V) and (π', V') are irreducible unitary representations of G.

(a) Let $L : V \to V'$ be any linear transformation. Show that

$$L' = \frac{1}{|G|} \sum_{g \in G} \pi'(g^{-1}) L \pi(g)$$

is an intertwining operator $L' : V \to V'$.

(b) If π and π' are inequivalent, apply Schur's Lemma and part (a) to the linear transformation $Lu = \langle u, v \rangle v'$ and consider $\langle L'u, u' \rangle$.

(c) Same as (b) with $V = V'$, but also consider $Trace(L')$ in two ways.

4. (a) Find orthonormal bases for $L^2(D_8)$ and $L^2(Q)$ using the irreducible representations.

(b) Apply Fourier's Trick and Parseval's Identity to $f = \delta_r + \delta_c$ in $L^2(D_8)$.

(c) Same as (b) for $f = 2\delta_i - \delta_j$ in $L^2(Q)$.

5. Let (π, V) be a representation of G. Suppose $V = \bigoplus_i V_i$ as an orthogonal direct sum of irreducible subrepresentations. Let n_σ be the multiplicity of the irreducible type σ in π. That is, n_σ is the number of V_i of type σ.

(a) Show that $\langle \chi_\pi, \chi_\sigma \rangle = n_\sigma$.

(b) Show that $\langle \chi_\pi, \chi_\pi \rangle = \sum_\sigma n_\sigma^2$.

(c) Show that π is irreducible if and only if $\langle \chi_\pi, \chi_\pi \rangle = 1$.

Date: August 6, 2012.

6. (a) Prove that $|\chi_\pi(g)| \leq dim(V)$ for all g in G as follows:

Suppose $\omega_1, \ldots, \omega_n$ are unit complex numbers. That is, each $\omega_k = e^{i\theta_k}$ for some real θ_k. Prove that

$$|\sum_k \omega_k| \leq n$$

with equality if and only if $\omega_1 = \cdots = \omega_n$.

(b) Suppose $|\chi_\pi(g)| = dim(V)$ for all irreducible representations (π, V) of G. Prove that g is in $Z(G)$.

(c) Verify by checking character tables until convinced. Explain the abelian case.

7. Let (π, V) be a nontrivial representation of G.

(a) Suppose $\chi_\pi(g) = dim(V)$. Prove that there exists a proper normal subgroup N of G containing g.

(b) Prove that π is faithful if and only if $\chi_\pi(g) = dim(V)$ implies $g = e$.

(c) Prove that G is simple if and only if for each nontrivial irreducible representation π, $\chi_\pi(g) = dim(V)$ implies $g = e$.

8. Let $G = S_4$, the symmetric group on 4 letters. G has five conjugacy classes, trivial center, and commutator subgroup A_4. S_4 has normal subgroups $\{e\}$, N $= \{(12)(34), (13)(24), (14)(23)\}$, A_4, and S_4. The first two rows of the character table are

G	e (1)	(12) (6)	(123) (8)	(1234) (6)	(12)(34) (3)
χ_{triv}	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1

(a) For the third row, find the irreducible types in the permutation representation $(\pi, \mathbb{C}^4)$ of S_4. Call the irreducible three-dimensional representation π_1.

(b) For the fourth row, determine the isomorphism class of G/N and extend a two-dimensional irreducible representation π_2 of G/N to G.

(c) Find the fifth row by considering $\chi_{sgn} \otimes \pi_1$.

(d) Which irreducible representations are faithful? How do they restrict to A_4?

(e) Determine the irreducible types with multiplicity in $\pi_1 \otimes \pi_1$.

9. Let (π, V) be the irreducible two-dimensional representation of S_3. Give a formula for the multiplicity of each irreducible type in $\pi \otimes \cdots \otimes \pi$ (n factors). Verify dimension counts.

10. Let G be the semidirect product of $\mathbb{Z}/3$ on $\mathbb{Z}/7$ defined by

$$y^3 = e,\ x^7 = e,\ yxy^{-1} = x^2.$$

G has 21 elements, trivial center, and commutator subgroup $\langle x \rangle$.

(a) With $\omega = e^{2\pi i/3}$, complete the following character table for G using orthonormality of rows and columns (Problem 1(a)).

G	e (1)	x (3)	x^{-1} (3)	y (7)	y^{-1} (7)
χ_0	1	1	1	1	1
χ_1	1	1	1	ω	ω^2
χ_2	1	1	1	ω^2	ω
π_1	a	b	c	d	e
π_2	m	n	p	q	r

(Hint: Consider the lengths of the last two columns first. Then note that $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$.)

(b) G may be realized as a subgroup of the affine motion group $H = Aff(\mathbb{Z}/7)$. That is, let $X = \mathbb{Z}/7$ and H the set of symmetries on X in the form $k \mapsto ak + b$ for a in $\mathbb{Z}/7 \setminus \{0\}$ and b in $\mathbb{Z}/7$. For G, we restrict a to values in $\{0, 2, 4\}$.

Let V be the complex vector space with basis $\{e_0, \dots, e_6\}$ and consider the permutation representation (π, V) of G induced by the group action on X. Determine the irreducible types that occur in π. Use x with $(a, b) = (1, 1)$ and y with $(a, b) = (2, 0)$.