INTRO TO GROUP REPS - AUGUST 13, 2012
PROBLEM SET 10
RT8. FINITE GROUPS 2

1. Let N be a normal subgroup of G, and suppose (π, V) is an irreducible representation of G/N. Recall that if $q : G \to G/N$ is the natural quotient homomorphism, then we obtain a representation (π', V) of G by defining $\pi' = \pi \circ q$. Give three proofs that π' is irreducible.

2. Let C_x denote the conjugacy class of x in S_3. Define e_x in $Class(S_3)$ by

$$e_x(g) = \begin{cases} 1 & g \in C_x \\ 0 & \text{otherwise} \end{cases}.$$

Express $e_e,\ e_{(12)},$ and $e_{(123)}$ in terms of the character basis for $Class(S_3)$, and verify Parseval's Identity.

3. Explain why the character tables for S_n and D_{2n} always consist of real entries. State a necessary condition for a character table to have non-real entries.

4. (a) Let (π, V) be an irreducible unitary representation of G and u, v in V. Prove that

$$\langle \phi_{u,v}, \chi_\pi \rangle = \frac{1}{d_\pi} \langle u, v \rangle$$

and

$$d_\pi \phi_{u,v} * \chi_\pi = d_\pi \chi_\pi * \phi_{u,v} = \phi_{u,v}.$$

(b) Verify associativity of convolution on $L^2(G)$.

(c) Verify that $f * h = h * f$ if h is in $Class(G)$. Also verify that $f * h$ is in $Class(G)$ when f, h are.

5. Let (π, V) be an irreducible representation of G with dimension greater than 1.

(a) If χ_π takes only integer values, show that $\chi_\pi(g) = 0$ for some g in G.

(b) This is true in general for irreducible π with dimension greater than 1. Verify for all character tables in the problem sets.

6. Let (π, V) be the irreducible representation of S_3 with dimension 2.

(a) Determine the irreducible subrepresentations in $\sigma = \pi \otimes \pi$.

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(b) Compute each projection operator $P_{\pi'} = d_{\pi'}\sigma(\overline{\chi_{\pi'}})$.

(c) Find bases for each subspace of a given irreducible type.

7. Let $(\pi, \mathbb{C}^n)$ be the usual representation of S_n by permuting the standard basis vectors. π decomposes into a trivial type and an irreducible $(n-1)$-dimensional type. Let $f(\sigma)$ denote the number of elements in $\{1, 2, \ldots, n\}$ fixed by σ.

(a) Prove that

$$n! = \sum_{\sigma \in S_n} f(\sigma)$$

and

$$2n! = \sum_{\sigma \in S_n} [f(\sigma)]^2.$$

(b) Verify for $n = 3, 4, 5$.

8. Suppose G acts on a set X, and let V be the vector space with basis $\{e_x \mid x \in X\}$. Let (π, V) be the permutation representation of G on V. Let $f(g)$ denote the number of elements in X fixed by g.

(a) Let n_σ be the multiplicity of the irreducible type σ in π, and let n_X be the number of orbits in X by G. Prove that

$$\sum_{g \in G} f(g) = n_X |G|.$$

and

$$\sum_{g \in G} [f(g)]^2 = (\sum_\sigma n_\sigma^2)|G|.$$

(b) Verify (a) directly for the usual action of D_{2n} on the vertices of a regular n-sided polygon. For tables, see Problems 10 in SS7 and SS8.

9. Same notation as Problem 7(a). Refer to SS4, Problem 6, and SS5, Problem 3. Using the projection formula $P_\sigma = d_\sigma \pi(\overline{\chi_\sigma})$, extract fixed point formulas from the (i, j)-th entries using both irreducible types.

10. Let M_σ be the span of the matrix coefficient of the irreducible representation (σ, V_σ). Give another proof of the Plancherel formula

$$\langle f, f \rangle = \sum_\sigma d_\sigma Trace(\sigma(f)\sigma(f)^*)$$

as follows:

(a) Show that $\sigma(\cdot) : L^2(G) \to Hom_{\mathbb{C}}(V_\sigma, V_\sigma)$ is an interwtwining operator between $L \otimes R$ and $\tilde{\sigma}$, where

$$[\tilde{\sigma}(g_1, g_2)]T(v) = \sigma(g_1)T\sigma(g_2^{-1})v.$$

(b) Explain why $(\tilde{\sigma}, Hom_{\mathbb{C}}(V_\sigma, V_\sigma))$ is irreducible as a representation of $G \times G$.

(c) Show that

$$\langle T_1, T_2 \rangle = Trace(T_1 T_2^*)$$

is a Hermitian inner product on $Hom_{\mathbb{C}}(V_\sigma, V_\sigma)$ invariant under $G \times G$.

(d) Apply both cases of Schur's Lemma to obtain the proof. Use $f = \chi_\sigma$ to determine scaling factors.

11. Suppose that $G = G_1 \times G_2$. If (π, V) (resp. (π', V')) is a representation of G_1 (resp. G_2), we define the outer tensor product representation $(\pi \otimes \pi', V \otimes V')$ on G by linearly extending

$$(\pi \otimes \pi')(g_1, g_2)(v \otimes w) = \pi(g_1)v \otimes \pi'(g_2)w.$$

(a) Prove that (σ, W) is an irreducible representation of G if and only if σ may be expressed as an outer tensor product of irreducible representations of G_1 and G_2.

(Not intended as a problem, but no convenient place to include the proof. See next Solution Set.)

(b) If $(\pi, \mathbb{C}^n)$ is an irreducible representation of G_1, prove that

$$W = Span_{\mathbb{C}}\{\pi(g) \mid g \in G\} = M(n, \mathbb{C}),$$

the set of all $n \times n$ matrices with complex entries.

(c) Verify (b) for the usual representation of D_{2n} on $\mathbb{C}^2$.