INTRO TO GROUP REPS - AUGUST 20, 2012
PROBLEM SET 11
RT9. BASIC TENSOR ANALYSIS

1. Consider the Vandermonde matrix $V = V(a_1, \ldots, a_n)$. V is $n \times n$, and the i-th row is $(1, a_1, a_1^2, \ldots, a_1^{n-1})$. For example, a 3×3 Vandermonde matrix $V(a_1, a_2, a_3)$ has the form

$$\begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{bmatrix}.$$

We prove that

$$det(V) = \prod_{i<j}(a_j - a_i)$$

as follows:

(a) Verify in the case where some $a_i = a_j$.

(b) Assume the a_i are distinct. Factor the polynomial $p(x) = det(a_1, \ldots, a_{n-1}, x)$ and prove the equality using induction.

2. Let $\omega = e^{2\pi i/n}$. The character table for $\mathbb{Z}/n$ equals $V(1, \omega, \ldots, \omega^{n-1})$.

(a) Prove that

$$\prod_{i<j} |\omega^j - \omega^i| = n^{n/2}$$

(b) Verify for $n = 2, 3, 4, 6$.

3. Let (π, V) be a representation of G. A result of Burnside/Molien states that, if π is faithful, every irreducible representation of G occurs in some $\otimes^N \pi$.

(a) Prove that, for some positive N, the trivial representation occurs in $\otimes^N \pi$. What if $N = 2$? Give an example where $N \neq 2$. (SS9, Problem 10)

(b) Prove that, if π is irreducible, then the dual representation π^* occurs in some $\otimes^N \pi$.

(c) Verify the general result for $\mathbb{Z}/n$.

4. Identify the irreducible types in $\wedge^2 \pi$ and $\odot^2 \pi$ for π the irreducible three-dimensional representation of A_4.

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5. Identify the irreducible types in $\wedge^2\pi$ and $\odot^2\pi$ for each irreducible representation of S_4.

6. The symmetric groups on 5 letters, S_5, has 7 conjugacy classes, each determined by cycle structure. Construct the character table of S_5 as follows:

(a) Find the (one-dimensional) characters and the irreducible subrepresentations of the permutation representation on $\mathbb{C}^5$. Denote the irreducible subrepresentation of dimension 4 by π_4.

(b) Decompose $\odot^2\pi_4$ to obtain an irreducible subrepresentation of dimension 5. Call it π_5.

(c) Consider $sgn \otimes \pi_4$ and $sgn \otimes \pi_5$.

(d) Decompose $\wedge^2\pi_4$ to obtain an irreducible subrepresentation of dimension 6. Call it π_6.

7. The alternating group on 5 letters, A_5, has 5 conjugacy classes. Three come from S_5 without change, but the class of 5 cycles splits in half. Representatives for each new class are given by (12345) and (13524). Note that A_5 is simple and that $(12345)^{-1} = (15432)$ is in the same class as (12345).

(a) Construct the character table of A_5 by restricting each character in the table for S_5 to A_5. What happens to sgn?

(b) A_5 is isomorphic to the group of rigid motions of a regular icosahedron. Verify that the induced representation on $\mathbb{R}^3$ corresponds to an irreducible representation of A_5.

(c) Conjecture for $cos(2\pi/5)$ and $cos(4\pi/5)$? Verify.

8. Consider $G = SL(3, \mathbb{Z}/2)$, the simple group of order 168. Let X be the seven point set of non-zero elements in $(\mathbb{Z}/2)^3$, and define a group action σ of G on X by $\sigma(g)x = gx$. Let V be the complex vector space with basis $B = \{e_x \mid x \in X\}$.

Refer to Problem/Solution Set 11, Problem 10 of the Group Theory course. Verify that the permutation representation (π, V) of G has character

$SL(3,\mathbb{Z}/2)$	e (1)	o2 (21)	o4 (42)	o7 (24)	o7i (24)	o3 (56)
π	7	3	1	0	0	1

and that $\pi = \chi_{triv} \oplus \pi_6$, where π_6 is irreducible.

Here each class is labeled by the order of its elements, and o7i contains the inverses of o7.

9. (a) Find two more irreducible characters of G by decomposing $\odot^2\pi_6$ and $\wedge^2\pi_6$.

(b) Use orthogonality of the rows and columns of the character table to find the remaining irreducible classes:

$SL(3,\mathbb{Z}/2)$	e (1)	o2 (21)	o4 (42)	o7 (24)	o7i (24)	o3 (56)
χ_{triv}	1	1	1	1	1	1
π_6	6	2	0	-1	-1	0
π_8	8	0	0	1	1	-1
π_7	7	-1	-1	0	0	1
σ_1	a	b	c	d	e	f
σ_2	x	y	z	u	v	w

10. Let $X = \{1, 2, 3, 4, 5, 6\}$. The symmetric group S_6 contains 12 subgroups isomorphic to S_5. Six arise as stabilizer subgroups for elements of X under the usual action of S_6. The other six subgroups act transitively on X under this action.

(a) Using PS10, Problem 8(a), determine the conjugacy classes (cycle structure only) of a special S_5 subgroup. Verify the second fixed-point formula.

(b) Identify the irreducible types under the permutation representation of a special S_5 on X.