INTRODUCTION TO GROUP REPRESENTATIONS
JUNE 11, 2012
SOLUTION SET 1

1. (a) $R(e)(x) = xe^{-1} = x$,

$$R(h_1)[R(h_2)(x)] = R(h_1)(xh_2^{-1}) = xh_2^{-1}h_1^{-1} = x(h_1h_2)^{-1} = R(h_1h_2)x,$$

(b) $L(e)(x) = ex = x$,

$$L(h_1)[L(h_2)(x)] = L(h_1)(h_2x) = (h_1h_2)x = L(h_1h_2)x,$$

(c) $D(e)(x) = exe^{-1} = x$,

$$\begin{aligned} D(h_1)[D(h_2)(x)] &= D(h_1)(h_2xh_2^{-1}) = h_1h_2xh_2^{-1}h_1^{-1} \\ &= (h_1h_2)x(h_1h_2)^{-1} = D(h_1h_2)x, \end{aligned}$$

(d) $J(e,e)(x) = exe^{-1} = x$,

$$\begin{aligned} J(h_1,h_2)[J(h_3,h_4)(x)] &= J(h_1,h_2)(h_3xh_4^{-1}) = h_1h_3xh_4^{-1}h_2^{-1} \\ &= (h_1h_3)x(h_2h_4)^{-1} = D(h_1h_3,h_2h_4)x. \end{aligned}$$

2. (a) Using trig identities, $R(\theta)R(\theta') = R(\theta+\theta') = R(\theta')R(\theta)$. So the set commutes.

Next, the characteristic polynomial of $R(\theta)$ is $p(x) = det(xI - R(\theta)) = x^2 - 2cos(\theta) + 1$, which has roots $\lambda = cos(\theta) \pm isin(\theta) = e^{\pm i\theta}$ by Euler's formula. Observation suggests that we check the vectors $(1, i)$ and $(1, -i)$ for the eigenvector property.

$$R(\theta)\begin{pmatrix}1\\ i\end{pmatrix} = \begin{pmatrix}cos(\theta) + isin(\theta)\\ -sin(\theta) + icos(\theta)\end{pmatrix} = e^{i\theta}\begin{pmatrix}1\\ i\end{pmatrix}.$$

A similar computation holds for $(1, -i)$, or conjugate both sides of the above equation.

(b) The corresponding hyperbolic trig identities imply that $H(t)H(t') = H(t+t') = H(t')H(t)$. So the set commutes.

Next, the characteristic polynomial of $H(t)$ is $p(x) = det(xI - H(t)) = x^2 - 2cosh(t) + 1$, which has roots $\lambda = cosh(t) \pm sinh(t) = e^{\pm t}$. Observation suggests that we check the vectors $(1, 1)$ and $(1, -1)$ for the eigenvector property.

$$H(t)\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}cosh(t) + sint(t)\\ sin(t) + cosh(t)\end{pmatrix} = e^{t}\begin{pmatrix}1\\ 1\end{pmatrix}.$$

A similar computation holds for $(1, -1)$.

Date: July 10, 2012.

3. (a) Note that A permutes the standard basis vectors: $e_1 \mapsto e_2 \mapsto e_3 \mapsto e_4 \mapsto e_1$. So $A^4 = I$. Now

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

One verifies Cayley-Hamilton directly from the following characteristic polynomials:

(1) $A: \ p(x) = x^4 - 1$,
(2) $A^2: \ p(x) = (x^2 - 1)^2$,
(3) $A^3: \ p(x) = x^4 - 1$.

(b) Over $\mathbb{C}$, we have the following minimal polynomials

(1) $A: \ p(x) = (x+1)(x-1)(x+i)(x-i)$,
(2) $A^2: \ p(x) = (x+1)(x-1)$,
(3) $A^3: \ p(x) = (x+1)(x-1)(x+i)(x-i)$.

Note that each minimal polynomial factors into distinct linear factors, confirming that each is diagonalizable. Since the A^k commute, it is enough to find an eigenbasis for A, which has one-dimensional eigenspaces. For each eigenvalue, we have column eigenvectors

(1) $1: \ (1, 1, 1, 1)$,
(2) $-1: \ (1, -1, 1, -1)$,
(3) $i: \ (1, -i, -1, i), .$
(4) $-i: \ (1, i, -1, -i)$.

The reader should verify the characteristic equation $A^k v = c^k v$ in each case. Compare with Problem 8(b).

4. (a) We verify the four group properties: suppose A, B in $GL(n, \mathbb{C})$.

(1) Closed: $det(AB) = det(A)det(B) \neq 0$ so AB is invertible if A and B are,
(2) Associative: property of matrix multiplication,
(3) Identity: $I = diag(1, \ldots, 1)$ satisfies $AI = IA = A$ for all (invertible) A, and
(4) Inverse: the inverse of A exists if and only if $det(A) \neq 0$, and then $det(A^{-1}) = 1/det(A) \neq 0$.

(b) First we show the subgroup properties: suppose A, B in $SL(n, \mathbb{C})$.

(1) Closure: $det(AB) = det(A)det(B) = 1$,
(2) Nonempty: $det(I) = 1$, so I is in $SL(n, \mathbb{C})$, and
(3) Inverse: if $det(A) = 1$, then $det(A^{-1}) = 1/det(A) = 1$.

To show the normal property, we need gng^{-1} in $SL(n, \mathbb{C})$ when n is in $SL(n, \mathbb{C})$ and g is in $GL(n, \mathbb{C})$. Now

$$det(gng^{-1}) = det(g)det(n)det(g^{-1}) = det(g)1det(g^{-1}) = 1.$$

5. (a-c) These are all straightforward properties of homomorphisms.

6. It is straightforward to check Problem 5 and the homomorphism property using the generators and relations in the notes for π_1:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

By observation, $\pi_2 = det(\pi_1)$, and we have that π_2 is a homomorphism by the multiplicative property of the determinant: $det(AB) = det(A)det(B)$.

7. Consider the permutation representation π on $\mathbb{C}^3$. Suppose (x, y, z) spans a one-dimensional subrepresentation. Then $\pi(\sigma)(x, y, z) = \lambda_\sigma(x, y, z)$ for scalar λ_σ. If σ has order 2, then $\lambda_\sigma = \pm 1$. Without loss of generality, suppose $\lambda_{(12)} = -1$. Then $(y, x, z) = (-x, -y, -z)$ implies that $z = 0$. Applying (13) and (23) gives $x = y = 0$.

Now if all $\lambda_\sigma = 1$, then $x = y = z$, which gives the trivial subrepresentation spanned by $(1, 1, 1)$.

8. (a) If $gh = hg$ the homomorphism property of π implies that

$$\pi(x)\pi(y) = \pi(xy) = \pi(yx) = \pi(y)\pi(x).$$

(b) For the cyclic groups, it is enough to determine where the generating element 1 goes:

$\mathbb{Z}/2$	0	1
χ_0	1	1
χ_1	1	-1

Let $\omega = e^{2\pi/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

$\mathbb{Z}/3$	0	1	2
χ_0	1	1	1
χ_1	1	ω	ω^2
χ_2	1	ω^2	ω

$\mathbb{Z}/4$	0	1	2	3
χ_0	1	1	1	1
χ_1	1	i	-1	-i
χ_2	1	-1	1	-1
χ_3	1	-i	-1	i

$\mathbb{Z}/2 \times \mathbb{Z}/2$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
χ_0	1	1	1	1
χ_1	1	-1	1	-1
χ_2	1	1	-1	-1
χ_3	1	-1	-1	1

(c) No. By observation, each has a nontrivial kernel. A two-dimensional faithful representation arises by taking, for instance, $\pi(g) = \begin{pmatrix} \chi_2(g) & 0 \\ 0 & \chi_3(g) \end{pmatrix}$.

9. Suppose an invariant subspace is spanned by (a, b). By invariance under G, we have $\pi(x)(a, b) = c_x(a, b)$ for all x in $\mathbb{R}$. Here c_x is a scalar dependent on x. On the other hand, we have directly that $\pi(x)(a, b) = (a + bx, b)$. So either $b = 0$ or $c_x = 1$ for all x. In the first case, we have the invariant subspace $\mathbb{C}(1, 0)$. In the second case, $(a + bx, b) = (a, b)$ for all x and and again we have that $b = 0$

10. (a) Let (σ, V) be a representation of G/N. Then we need that $\pi = \sigma \circ q$ is a homomorphism from G to $GL(n, V)$. Now $q : G \to G/N$ defined by $q(g) = gN$ is a homomorphism with kernel N.

We show π is a homomorphism: if g, h are in G, then $\pi(gh) = \pi(g)\pi(h)$. Now

$$\pi(gh) = \sigma(q(gh)) = \sigma(ghN) = \sigma(gNhN) = \sigma(gN)\sigma(hN) = \pi(g)\pi(h).$$

(b) D_8 has three subgroups of order 4, which are normal by the Index Two theorem. In each case, the quotient G/N is a a two element group, and we get a non-trivial homomorphism by sending $N \mapsto 1, \ gN \mapsto -1$.

D_8	e	r	r^2	r^3	c	cr	cr^2	cr^3
χ_1	1	1	1	1	-1	-1	-1	-1
χ_2	1	-1	1	-1	-1	1	-1	1
χ_3	1	-1	1	-1	1	-1	1	-1

(c) Same idea as Part (b):

Q	± 1	$\pm i$	$\pm j$	$\pm k$
χ_1	1	1	-1	-1
χ_2	1	-1	1	-1
χ_3	1	-1	-1	1

11. (a) As seen in 10(a), the composition of homomorphisms is another homomorphism. Here $G \to G \to GL(n, \mathbb{C})$.

(b) Under the inner automorphism $i_x(g) = xgx^{-1}$, we have $\pi \circ i_x : G \to \mathbb{C}^*$, so

$$(\pi \circ i_x)(g) = \pi(xgx^{-1}) = \pi(x)\pi(g)\pi(x)^{-1} = \pi(g)$$

since $\mathbb{C}^*$ is abelian under multiplication.

(c) To show a group action, we must show

(1) $I(\pi) = \pi : I(\pi) = \pi \circ I^{-1} = \pi \circ I = \pi$, and
(2) $(\sigma_1 \circ \sigma_2)(\pi) = \sigma_1(\sigma_2(\pi))$:

$$\sigma_1(\sigma_2(\pi)) = \sigma_1(\pi \circ \sigma_2^{-1}) = \pi \circ \sigma_2^{-1} \circ \sigma_1^{-1} = \pi \circ (\sigma_1 \circ \sigma_2)^{-1} = (\sigma_1 \circ \sigma_2)(\pi).$$

(d) For Problem 8(b), the trivial representation is unchanged by automorphisms, so $\{\chi_0\}$ is always an orbit.

For $\mathbb{Z}/2$, $Out(G)$ is trivial, so both one-dimensional representations are unchanged and $\{\chi_1\}$ is the other orbit.

For $\mathbb{Z}/3$, $Out(G) \cong \mathbb{Z}/2$, with nontrivial element represented by multiplication by 2. This switches χ_1 and χ_2, so $\{\chi_1, \chi_2\}$ is the other orbit.

For $\mathbb{Z}/4$, $Out(G) \cong \mathbb{Z}/2$, with nontrivial element represented by multiplication by 3. This switches χ_1 and χ_3, so the other orbits are $\{\chi_1, \chi_3\}$ and $\{\chi_2\}$.

For $\mathbb{Z}/2 \times \mathbb{Z}/2$, $Out(G) \cong SL(2, \mathbb{Z}/2) \cong S_3$, acting by permuting bases of G as a vector space over $\mathbb{Z}/2$. The other orbit is $\{\chi_1, \chi_2, \chi_3\}$.

For D_8, $Out(G) \cong \mathbb{Z}/2$ and is represented by $r \mapsto r,\ c \mapsto cr$. This automorphism fixes $\langle r \rangle$ and interchanges the other two four-element subgroups (the kernels). Thus the group action fixes the first representation and switches the other two. So there are two orbits, $\{\chi_1\}$ and $\{\chi_2, \chi_3\}$.

For Q, $Out(G) \cong S_3$ by permuting the sets $\{\pm i\}, \{\pm j\}$, and $\{\pm k\}$. The representations are permuted according to the effect on the kernel. This group action is transitive, so one orbit with three elements.