INTRO TO REP THEORY - JUNE 18, 2012
SOLUTION SET 2
RT2. UNITARY REPRESENTATIONS

1. (a) This is a straight-forward check using the definitions.

(b) First check that $(Lw)^* = w^* L^*$; note that this can be separated into n row-column products. Also note that $(L^*)^* = L$. Then we have that

$$\langle Lv, w\rangle = w^*(Lv) = w^*(L^*)^* v = (L^* w)^* v = \langle v, L^* w\rangle.$$

(c) For v, w in $\mathbb{C}^n$,

$$\langle L_1 L_2 v, w\rangle = \langle L_2 v, L_1^* w\rangle = \langle v, L_2^* L_1^* w\rangle.$$

But this expression also equals $\langle v, (L_1 L_2)^* w\rangle$, so $(L_1 L_2)^* = L_2^* L_1^*$.

Just in case: if $\langle Av, w\rangle = \langle Bv, w\rangle$ for all v, w in $\mathbb{C}^n$, then $A = B$. Subtracting gives the equivalent statement: if $\langle Xv, w\rangle = 0$ for all v, w in $\mathbb{C}^n$, then $X = 0$. To show this note that the (i, j)-th entry of X is equal to $\langle Xe_j, e_i\rangle$. (Note the reversed order; verify with your favorite 2×2 matrix.)

2. (a) Noting to conjugate the second vector:

(1) $||v_1||^2 = 1/\sqrt{2} \cdot 1/\sqrt{2} + i/\sqrt{2} \cdot -i/\sqrt{2} = 1$,
(2) $||v_2||^2 = 1/\sqrt{2} \cdot 1/\sqrt{2} + -i/\sqrt{2} \cdot i/\sqrt{2} = 1$, and
(3) $\langle v_1, v_2\rangle = 1/\sqrt{2} \cdot 1/\sqrt{2} + i/\sqrt{2} \cdot i/\sqrt{2} = 0$.

(b) Again

(1) $c_1 = \langle v, v_1\rangle = i \cdot 1/\sqrt{2} + i \cdot -i/\sqrt{2} = (1 + i)/\sqrt{2}$, and
(2) $c_2 = \langle v, v_2\rangle = i \cdot 1/\sqrt{2} + i \cdot i/\sqrt{2} = (-1 + i)/\sqrt{2}$.

Now

$$c_1 v_1 + c_2 v_2 = \frac{1+i}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix} + \frac{-1+i}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix} = \begin{bmatrix} i \\ i \end{bmatrix} = v,$$

and

$$|c_1|^2 + |c_2|^2 = 1 + 1 = 2; \qquad ||v||^2 = 1 + 1 = 2.$$

3. We find the parallel direction first and then subtract it off v to get the orthogonal direction. For the parallel direction, we use the formula $v_{||} = \frac{\langle v,u\rangle}{\langle u,u\rangle} u$.

Date: June 15, 2012.

Here $u = (1+i, -i)$. and $v = (2, i)$. Substitution gives $||u||^2 = 3$ and $\langle v, u\rangle = 1 - 2i$. So

$$v_{||} = (1 - i/3, -2/3 - i/3) \quad \text{and} \quad v_\perp = v - v_{||} = (1 + i/3, 2/3 + 4i/3).$$

Immediately $v = v_{||} + v_\perp$ and one has that

$$\langle v_{||}, v_\perp\rangle = (1 - i/3)\cdot(1 - i/3) + (-2/3 - i/3)\cdot(2/3 - 4i/3) = 8/9 - 2i/3 - 8/9 + 6i/9 = 0.$$

4. (a) (1) is equivalent to (2): First, if U is unitary,

$$\langle Uv, Uw\rangle = \langle v, U^*Uw\rangle = \langle v, Iw\rangle = \langle v, w\rangle.$$

Conversely, if $\langle Uv, Uw\rangle = \langle v, w\rangle$ for all v, w, then $\langle v, (U^*U - I)w\rangle = 0$ for all v, w, which we have seen above means that $U^*U - I = 0$ or $U^*U = I$. Thus $U^* = U^{-1}$ and $UU^* = I$ also.

(1) is equivalent to (3): rewrite all row-column products in $U^*U = I$, and

(1) is equivalent to (4): rewrite all row-column products in $UU^* = I$.

(b) We verify the subgroup properties: suppose A, B in $U(n)$.

(1) Closure: $AB(AB)^* = ABB^*A^* = AIA^* = I$,
(2) Inverses: since $I = AA^{-1}$, $I = (A^{-1})^*A^* = (A^{-1})^*A^{-1}$. Thus A^{-1} is in $U(n)$,
(3) Nonempty: the identity matrix I satisfies $II^* = I$.

If A is in $O(n)$, then $A^TA = I$. Since A has real entries, $A^T = A^*$ and $A^*A = I$.

(c) We verify the subgroup properties: suppose A, B in $SU(n)$.

(1) Closure: AB is in $U(n)$ and $det(AB) = det(A)det(B) = 1$,
(2) Inverses: A^{-1} is in $U(n)$ and $det(A^{-1}) = 1/det(A) = 1$,
(3) Nonempty: the identity matrix I is in $U(n)$ and $det(I) = 1$.

For the normal property, if g is in $U(n)$ and A is in $SU(n)$, then gAg^{-1} is in $U(n)$ and $det(gAg^{-1}) = det(A) = 1$.

5. (a) Let v be an (nonzero) eigenvector with eigenvalue λ. Then

$$\lambda\langle v, v\rangle = \langle Xv, v\rangle = \langle v, Xv\rangle = \overline{\lambda}\langle v, v\rangle.$$

Since $\langle v, v\rangle > 0$, $\lambda = \overline{\lambda}$, and λ is real.

(b) Let v (resp. w) be an (nonzero) eigenvector with eigenvalue λ (resp. μ). Then

$$\lambda\langle v, w\rangle = \langle Xv, w\rangle = \langle v, Xw\rangle = \overline{\mu}\langle v, w\rangle.$$

Since μ is real, $(\lambda - \mu)\langle v, w\rangle = 0$, and, since $\lambda \neq \mu$, $\langle v, w\rangle = 0$.

(c) We show that if v is in $W^\perp$ then Xv is also in $W^\perp$. That is, if $\langle v, w\rangle = 0$ for all w in W, then $\langle Xv, w\rangle = 0$ for all w in W also. Now if w is in W, then invariance means that Xw is in W and

$$\langle Xv, w\rangle = \langle v, Xw\rangle = 0$$

since v is in $W^\perp$.

(d) We proceed by induction on the dimension of V. If V has dimension one, then Hermitian means X is a real scalar and we get an orthonormal basis by choosing a unit vector in V.

For the induction step, suppose the result is true when the dimension is less than n and that the dimension of V is less than $n+1$. By part (a), there exists at least one real eigenvalue with unit eigenvector v. Let W be the span of v. Since W is invariant under X, so $W^{\perp}$ is invariant under X also by (c). This means $X : W^{\perp} \to W^{\perp}$. Since the Hermitian condition holds on $W^{\perp}$, the induction hypothesis is satisfied, and we can find an orthonormal basis B of $W^{\perp}$ consisting of eigenvectors with real eigenvalues. Now $B \cup \{v\}$ is the basis that we seek for V.

(e) First $p_X(x) = det(xI - X) = x^2 - 1$, so X has eigenvalues ± 1. If $\lambda = 1$, then

$$Null(X - I) = Null \begin{bmatrix} -1 & i \\ -i & -1 \end{bmatrix} = Null \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}.$$

Thus $v_1 = (1/\sqrt{2}, -i/\sqrt{2})$ is an unit eigenvector with eigenvalue 1.

Let's find a unit eigenvector for $\lambda = -1$ indirectly. We have $Xv_1 = v_1$, so conjugating both sides of the equation gives $\overline{X}\overline{v_1} = \overline{v_1}$. But $\overline{X} = -X$, so $X\overline{v_1} = -\overline{v_1}$. Thus

$$v_2 = \overline{v_1} = (1/\sqrt{2}, i/\sqrt{2})$$

is a unit eigenvector with eigenvalue -1. Note that $\langle v_1, v_2 \rangle = 0$, confirming (b).

We define $P = [v_1 v_2]$, and the diagonalization formula states that $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = P^{-1}XP$.

6. (a) Let v be an (nonzero) eigenvector with eigenvalue λ. Then

$$\lambda\overline{\lambda}\langle v, v \rangle = \langle Uv, Uv \rangle = \langle v, U^*Uv \rangle = \langle v, v \rangle.$$

Since $\langle v, v \rangle > 0$, $|\lambda| = 1$.

(b) Let v (resp. w) be an (nonzero) eigenvector with eigenvalue λ (resp. μ). Then

$$\lambda\overline{\mu}\langle v, w \rangle = \langle Uv, Uw \rangle = \langle v, U^*Uw \rangle = \langle v, w \rangle.$$

Thus $(\lambda\overline{\mu} - 1)\langle v, w \rangle = 0$, and, since $\lambda \neq \mu$, $\langle v, w \rangle = 0$.

(c) Since A is invertible, A^{-1} exists. Since W is invariant under A, then A sends W to W isomorphically, and A^{-1} carries W to W. That is, if $Av = w$ then $A^{-1}w = v$. Thus W is invariant under A^{-1}.

If U is unitary, then $U^* = U^{-1}$ and we adapt the argument of Problem 5(c).

(d) Same proof as Problem 5(d) with appropriate adjustments.

(e) First $p_U(x) = det(xI - U) = x^2 + 1$, so U has eigenvalues $\pm i$. If $\lambda = i$, then

$$Null(X - iI) = Null \begin{bmatrix} -i & i \\ i & -i \end{bmatrix} = Null \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Thus $v_1 = (1/\sqrt{2}, 1/\sqrt{2})$ is an unit eigenvector with eigenvalue i.

One sees that the trick in Problem 5(e) does not apply since v_1 is real. Instead we expect the remaining eigenspace to be orthogonal to v_1, so we let $v_2 = (1/\sqrt{2}, -1/\sqrt{2})$ and verify the characteristic equation $Av_2 = -iv_2$ directly.

7. We see that A has eigenvalues 2 and -4 with unit eigenvectors

$$v_1 = (\sqrt{2}/2, \sqrt{2}/2) \quad \text{and} \quad v_2 = (\sqrt{2}/2, -\sqrt{2}/2)$$

respectively. Let $P = [v_1\ v_2]$. Then

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix} = P^{-1}AP.$$

For the projection P_2 onto $E_2 = \mathbb{C}v_1$, we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = P^{-1}P_2P \quad \text{or} \quad P_2 = P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}P^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

For the other eigenspace E_{-4}, a similar computation gives $P_{-4} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$.

The verifications for the P's are straightforward.

We have that $P_2e_1 = (1/2, 1/2)$ and $P_{-4}e_1 = (1/2, -1/2)$.

We also have $Ae_1 = (-1, 3) = 2(1/2, 1/2) + (-4)(1/2, -1/2)$.

8. (a) $\pi(k)\pi(l) = A^k \cdot A^l = A^{k+l} = \pi(k + l)$.

(b) (1) if A is diagonalizable, then the A^k commute and are simultaneously diagonalizable. Thus there exists a basis in which π maps to powers of a diagonal matrix with non-zero diagonal entries.

(2) A is diagonalizable with real eigenvalues, so same as (1) with real entries, and

(3) A is diagonalizable with unit eigenvalues, so same as (1) with unit diagonal entries,

If the kernel of π is nontrivial, then π quotients to a representation of $\mathbb{Z}/n$ and each $\pi(g)^n = I$. Since $p(x) = x^n - 1$ has distinct roots, the minimal polynomial of $\pi(g)$ factors into distinct linear factors, and $\pi(g)$ is diagonalizable. Since the eigenvalues have $|\lambda| = 1$, (1) and (3) hold.

9. (a) With respect to the standard basis, $\pi(r) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\pi(c) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Since D_8 is generated by r and c, and the product of unitary matrices is unitary, each $\pi(g)$ is unitary.

If there were a one-dimensional subrepresentation spanned by (x, y), then

$$\pi(g)(x, y) = \lambda_g(x, y)$$

for some scalar λ_g. Since $\pi(c)^2 = I$, $\lambda_c = \pm 1$ and (x, y) is zero or an eigenvector for $\pi(c)$. That is, either a multiple of e_1 or e_2 if nonzero. Since $\pi(r)e_1 = -e_2$ and $\pi(r)e_2 = e_1$, we must have $x = y = 0$.

(b) $\pi(c)$ is diagonal, so $B_1 = \{e_1, e_2\}$.

Since $\pi(r)^2 = -I$, the eigenvalues of $\pi(r)$ are $\pm i$ with associated unit eigenvectors $(1/\sqrt{2}, -i/\sqrt{2})$ and $(1/\sqrt{2}, i/\sqrt{2})$.

10. (a) Same as Problem 9(a), but $\pi(r) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$.

(b) $\pi(c)$ is diagonal, so $B_1 = \{e_1, e_2\}$.

Since $\pi(r)^3 = I$ but 1 is not an eigenvalue (no fixed vectors), the eigenvalues are ω and ω^2, where $\omega = e^{2i\pi/3}$. Noting that $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, one finds associated unit eigenvectors $(1/\sqrt{2}, -i/\sqrt{2})$ and $(1/\sqrt{2}, i/\sqrt{2})$.

11. (a) Since $\pi(1, 0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, we see that

$$\langle e_1, e_2 \rangle_1 = \langle X e_1, e_2 \rangle = 1,$$

but

$$\langle \pi(1, 0)e_1, \pi(1, 0)e_2 \rangle_1 = \langle X\pi(1, 0)e_1, \pi(1, 0)e_2 \rangle = -1.$$

Thus $\langle \cdot, \cdot \rangle_1$ is not invariant under G.

We compute each $\pi(g)^* X \pi(g)$: we have

$$\pi(0, 0) = I, \quad \pi(1, 0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \pi(0, 1) = -I, \quad \pi(1, 1) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

so

$$\frac{1}{4} \sum_g \pi(g)^* X \pi(g) = X + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + X + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 2I.$$

Since $\langle \cdot, \cdot \rangle$ is invariant under G, so is $\langle \cdot, \cdot \rangle_2$.

(b) The first part holds since $\pi(2) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ also. We compute each $\pi(g)^* X \pi(g)$:

$$\pi(0) = I, \quad \pi(1) = \begin{bmatrix} -1 & 0 \\ 0 & -i \end{bmatrix}, \quad \pi(2) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \pi(3) = \begin{bmatrix} -1 & 0 \\ 0 & i \end{bmatrix},$$

so

$$\frac{1}{4} \sum_g \pi(g)^* X \pi(g) = X + \begin{bmatrix} 2 & i \\ -i & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -i \\ i & 2 \end{bmatrix} = 2I.$$

Since $\langle \cdot, \cdot \rangle$ is invariant under G, so is $\langle \cdot, \cdot \rangle_2$.