INTRO TO GROUP REPS - JULY 2, 2012
SOLUTION SET 4
RT4. CONSTRUCTIONS FROM LINEAR ALGEBRA

1. (a) First $L_1^{-1} : V' \to V$ is a vector space isomorphism. Since $\pi'(g)L_1 = L_1\pi(g)$ for all g in G, we have that $L_1^{-1}\pi'(g) = \pi(g)L_1^{-1}$, and L_1^{-1} is an intertwining operator. Thus L_1^{-1} is an equivalence.

(b) If L_1 and L_2 are vector space isomorphisms, then so is the composition $L_2 \circ L_1$. Now for all g in G,

$$\pi''(g)(L_2 \circ L_1)(v) = \pi''(g)(L_2(L_1 v)) = L_2(\pi'(g)(L_1 v)) = L_2(L_1(\pi(g)v)),$$

so $\pi''(g)(L_2 \circ L_1) = (L_2 \circ L_1)\pi(g)$ and $L_2 \circ L_1$ is an intertwining operator. Thus it is also an equivalence.

2. (a) Reflexive: $(\pi, V) \sim (\pi, V)$ using $L = I$.

Symmetric: $(\pi, V) \sim (\pi', V')$ implies $(\pi', V') \sim (\pi, V)$. If $L : V \to V'$ is an equivalence, then $L^{-1} : V' \to V$ is also.

Transitive: if $(\pi, V) \sim (\pi', V')$ and $(\pi', V') \sim (\pi'', V'')$ then $(\pi, V) \sim (\pi'', V'')$. Suppose $L_1 : V \to V'$ and $L_2 : V' \to V''$ are equivalences. Then $L_2 \circ L_1 : V \to V''$ is also.

(b) We need to show in each case in (a) that $\langle Lv, Lw\rangle' = \langle v, w\rangle$ for all v and w.

Reflexive: $\langle Iv, Iw\rangle = \langle v, w\rangle$.

Symmetric: we assume that L is a unitary equivalence, so $\langle Lv, Lw\rangle' = \langle v, w\rangle$. That is $LL^* = L^*L = I$ or $L^* = L^{-1}$. Now

$$\langle L^{-1}v, L^{-1}w\rangle = \langle v, (L^{-1})^* L^{-1} w\rangle' = \langle v, (L^*)^* L^{-1} w\rangle' = \langle v, w\rangle'.$$

Transitive: we assume L_1 and L_2 are unitary equivalences. Then

$$\langle L_2(L_1 v), L_2(L_1 w)\rangle'' = \langle L_1 v, L_1 w\rangle' = \langle v, w\rangle.$$

(c) First, for the action on representations, we have $I(\pi) = \pi \circ I^{-1} = \pi$, and, for σ_1, σ_2 in $Aut(G)$,

$$(\sigma_1\sigma_2)(\pi)(g) = \pi((\sigma_1\sigma_2)^{-1}g) = \pi(\sigma_2^{-1}\sigma_1^{-1}g) = (\sigma_2\pi)(\sigma_1^{-1}g) = [\sigma_1(\sigma_2\pi)](g).$$

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To show the action passes to equivalence classes, we must show that if $(\pi, V) \sim (\pi', V')$, then $(\sigma\pi, V) \sim (\sigma\pi', V')$. Suppose $L : V \to V'$ is an equivalence. Then

$$L(\sigma\pi)(g) = L\pi(\sigma^{-1}(g)) = \pi'(\sigma^{-1}(g))L = (\sigma\pi')(g)L$$

since $\sigma^{-1}g$ is an element of G.

We have seen that if σ is an inner automorphism, then $\sigma\pi$ and π are equivalent. Thus the action passes to the quotient $Out(G) = Aut(G)/Inn(G)$.

3.

$$\begin{aligned}\text{(a) } \pi_{\oplus}(gh)(v, v') &= (\pi(gh)v, \pi'(gh)v') = (\pi(g)\pi(h)v, \pi'(g)\pi'(h)v') \\ &= \pi_{\oplus}(g)\pi_{\oplus}(h)(v, v'),\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad \pi^*(gh)v^*(w) &= v^*(\pi((gh)^{-1})(w) = v^*(\pi(h)^{-1}\pi(g)^{-1}w) \\ &= (\pi^*(h)v^*)(\pi(g)^{-1}w) = [\pi^*(g)(\pi^*(h)v^*)](w),\end{aligned}$$

$$\begin{aligned}\text{(c)} \quad [\sigma(gh)L](v) &= \pi'(gh)[L(\pi((gh)^{-1})v)] = \pi'(g)\pi'(h)[L(\pi(h)^{-1}\pi(g)^{-1}v)] \\ &= \pi'(g)[\sigma(h)L](\pi^{-1}(g)v) = \sigma(g)[\sigma(h)L](v).\end{aligned}$$

(d) Same as (a), then extend to general tensors linearly.

4. We leave sesquilinearity and Hermitian to the reader in most cases below.

(a) Inner product: $\langle (v, v'), (v, v') \rangle = \langle v, v \rangle + \langle v', v' \rangle' \geq 0$, and $= 0$ if and only if $v = 0$, $v' = 0$.

Invariance:

$$\begin{aligned}\langle \pi_{\oplus}(g)(v, v'), \pi_{\oplus}(g)(w, w') \rangle &= \langle (\pi(g)v, \pi'(g)v'), (\pi(g)w, \pi'(g)w') \rangle \\ &= \langle \pi(g)v, \pi(g)w \rangle + \langle \pi'(g)v', \pi'(g)w' \rangle' = \langle v, w \rangle + \langle v', w' \rangle' \\ &= \langle (v, v'), (w, w') \rangle.\end{aligned}$$

(b) Inner product: Straightforward. Note that reversing the order allows scalars to factor out correctly.

Invariance:

$$\begin{aligned}\langle \pi^*(g)\langle \cdot, v \rangle, \pi^*(g)\langle \cdot, w \rangle \rangle_* &= \langle \langle \pi(g)^{-1} \cdot, v \rangle, \langle \pi(g)^{-1} \cdot, w \rangle \rangle_* \\ &= \langle \langle \cdot, \pi(g)v \rangle, \langle \cdot, \pi(g)w \rangle \rangle_* = \langle \pi(g)w, \pi(g)v \rangle_* \\ &= \langle w, v \rangle = \langle \langle \cdot, v \rangle, \langle \cdot, w \rangle \rangle_*\end{aligned}$$

(c) Inner product: Note that if we choose orthonormal bases to obtain associated matrix $M_{L_1} = [c_{i,j}]$ for L_1, then $\langle L_1, L_1 \rangle = \sum |c_{i,j}|^2$.

Invariance:

$$\begin{aligned}\langle \sigma(g)L_1, \sigma(g)L_2\rangle &= \langle \pi'(g)L_1\pi(g)^{-1}, \pi'(g)L_2\pi(g)^{-1}\rangle \\ &= Trace(\pi'(g)L_1\pi(g)^{-1}(\pi'(g)L_2\pi(g)^{-1})^*) \\ &= Trace(\pi'(g)L_1\pi(g)^{-1}\pi(g)L_2^*\pi'(g)^{-1} \\ &= Trace(\pi(g)L_1L_2^*\pi(g)^{-1}) = Trace(L_1L_2^*) = \langle L_1, L_2\rangle.\end{aligned}$$

(d) Inner product: verify on the orthonormal basis $\{e_i \otimes f_i\}$.

Invariance: we show on monomials and extend linearly

$$\begin{aligned}\langle \pi\otimes\pi'(g)(v\otimes v'), \pi\otimes\pi'(g)(w\otimes w')\rangle &= \langle \pi(g)v\otimes\pi'(g)v', \pi(g)w\otimes\pi'(g)w'\rangle \\ &= \langle \pi(g)v, \pi(g)w\rangle\langle\pi'(g)v', \pi'(g)w'\rangle' \\ &= \langle v, w\rangle\langle v', w'\rangle' = \langle v\otimes w, v'\otimes w'\rangle.\end{aligned}$$

5. (a) We must show that $cf_1 + f_2 : V \to \mathbb{C}$ is linear:

$$\begin{aligned}(cf_1+f_2)(av+w) &= (cf_1)(av+w) + f_2(av+w) \\ &= c(af_1(v) + f_1(w)) + af_2(v) + f_2(w) \\ &= a(cf_1+f_2)(v) + (cf_1+f_2)(w).\end{aligned}$$

(b) The rule is $(\pi^*(g)v^*)(v') = v^*(\pi(g)^{-1}v)$. Note that

$$\pi(g)^*e_1^*(v) = \langle \pi(g)^{-1}v, e_1\rangle = \langle v, \pi(g)e_1\rangle.$$

Thus

(1) $\pi^*(e)e_1^* = e_1^*$,
(2) $\pi^*(12)e_1^* = \langle\cdot, \pi(12)e_1\rangle = e_2^*$,
(3) $\pi^*(13)e_1^* = \langle\cdot, \pi(13)e_1\rangle = e_3^*$,
(4) $\pi^*(23)e_1^* = \langle\cdot, \pi(23)e_1\rangle = e_1^*$,
(5) $\pi^*(123)e_1^* = \langle\cdot, \pi(123)e_1\rangle = e_2^*$, and
(6) $\pi^*(132)e_1^* = \langle\cdot, \pi(132)e_1\rangle = e_3^*$.

(c) $Lv = [2\ 1\ -2]v$.
(1) $\pi^*(23)Lv = [2\ 1\ -2]\pi(23)^{-1}v = [2\ -2\ 1]v$, and
(2) $\pi^*(123)Lv = [2\ 1\ -2]\pi(123)^{-1}v = [-2\ 2\ 1]v$.
Since we rearrange coordinates, the lengths of 3 are unchanged.

6. (a) Recall that the orthogonal projection onto W is defined as $P_W(v) = v_1$ where $v = v_1 + v_2$ is represented uniquely with v_1 in W and v_2 in $W^\perp$. We need to show that

$$P_W\pi(g)(v) = \pi(g)P_W(v).$$

Since both (π, W) and $(\pi, W^\perp)$ are subrepresentations, $\pi(g)v = \pi(g)v_1 + \pi(g)v_2$ represents $\pi(g)v$ as a sum with $\pi(g)v_1$ in W and $\pi(g)v_2$ in $W^\perp$. Thus

$$\pi(g)(P_W v) = \pi(g)v_1 = P_W(\pi(g)v).$$

(b) Define $L : W \to V/W^\perp$ by $Lw = w + W^\perp$. Since $V = W \oplus W^\perp$, W and $V/W^\perp$ have the same dimension. If $Lw = 0$ then w is in $W^\perp$, and, since $W \cap W^\perp = \{0\}$, $w = 0$. Thus L is a vector space isomorphism. Now

$$\pi(g)Lw = \pi(g)(w + W^\perp) = \pi(g)w + \pi(g)W^\perp = \pi(g)w + W^\perp = L(\pi(g)w),$$

so L is an intertwining operator.

(c) To find P_W with respect to the standard basis, we compute

$$P_W e_i = \frac{\langle e_i, (1,1,1)\rangle}{||(1,1,1)||^2}(1,1,1) = (1/3, 1/3, 1/3)$$

in each case. Since $P_W + P_{W^\perp} = I$, we have

$$P_W = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \qquad P_{W^\perp} = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}.$$

We leave it to the reader to show directly that $\pi(g)P_W = P_W\pi(g)$ for $g = (23), (123)$. These elements generate S_3. On the other hand, multiplying P_W by a permutation matrix leaves the entries unchanged, so we have in fact that $\pi(g)P_W = P_W = P_W\pi(g)$. Then note that $P_{W^\perp} = I - P_W$, both terms which commute with each $\pi(g)$.

7. (a) One-one: we show that $Ker(L) = 0$. Choose bases $B = \{v_j\}$ and $C = \{w_i\}$ for V and W. Let $B^* = \{v_j^*\}$ be the basis for V^* dual to B. Suppose $t = \sum_{i,j} c_{i,j} v_j^* \otimes w_i$ is in $Ker(L)$. Then

$$(Lt)(v_j) = c_{i,j}w_i = 0 \quad \text{and} \quad c_{i,j} = 0 \quad \text{for all } i, j.$$

Onto: Note that $T_{i,j}$ is a basis for $Hom_{\mathbb{C}}(V, W)$, where

$$T_{i,j}(v_k) = \begin{cases} w_i & j = k \\ 0 & \text{otherwise} \end{cases}.$$

But L is linear and $L(v_j^* \otimes w_i) = T_{i,j}$.

Thus L is a vector space isomorphism.

Next, on monomials,

$$\begin{aligned} [\pi(g)L(v^* \otimes w)](v') &= \pi'(g)[L(v^* \otimes w)](\pi(g)^{-1}v') \\ &= v^*(\pi(g)^{-1}v')\pi'(g)w \\ &= L[\pi^*(g)v^* \otimes \pi'(g)w](v') = L[(\pi_\otimes(g))(v^* \otimes w)](v'). \end{aligned}$$

(b) Suppose B and C are orthonormal bases. Using bilinearity, it is enough to show that $\{v_j^* \otimes w_i\}$ and $\{T_{i,j}\}$ are orthonormal bases of $V^* \otimes W$ and $Hom_{\mathbb{C}}(V, W)$ respectively.

Using the norm on tensors,

$$\langle v_j^* \otimes w_i, v_j^* \otimes w_i \rangle = \langle v_j^*, v_j^* \rangle_* \langle w_i, w_i \rangle = 1$$

since the dual basis is also orthonormal. On the other hand,

$$\langle T_{i,j}, T_{i,j} \rangle = Trace(T_{i,j} T_{i,j}^*) = 1;$$

with respect to the bases B and C, the associated matrix for $T_{i,j}$ has all zero entries save for a 1 in the (i, j)-th entry.

(c)

$$[\sigma(23)T](x,y,z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

so $[\sigma(23)]T(x, y, z) = (2x + y, x - y - z, x + z)$.

$$[\sigma(123)T](x,y,z) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

so $[\sigma(123)]T(x, y, z) = (-x + y + z, x + 2y, y - z)$.

8. It is enough to verify (a) - (c) on generators (23) and (123). First note that
   (1) $\pi(23)v_1 = v_2$,
   (2) $\pi(23)v_2 = v_1$, and
   (3) $\pi(23)v_3 = -v_3$.

   (1) $\pi(123)v_1 = v_3$,
   (2) $\pi(123)v_2 = -v_1$, and
   (3) $\pi(123)v_3 = -v_2$.

(a) $\pi(23)(v_1 \otimes v_2 - v_2 \otimes v_1) = v_2 \otimes v_1 - v_1 \otimes v_2 = sgn(23)(v_1 \otimes v_2 - v_2 \otimes v_1)$,

$$\begin{aligned} \pi(123)(v_1 \otimes v_2 - v_2 \otimes v_1) &= -v_3 \otimes v_1 + v_1 \otimes v_3 = -(v_2 - v_1) \otimes v_1 + v_1 \otimes (v_2 - v_1) \\ &= sgn(123)(v_1 \otimes v_2 - v_2 \otimes v_1). \end{aligned}$$

(b) $\pi(23)(v_1 \otimes v_1 + v_2 \otimes v_2 + v_3 \otimes v_3) = v_2 \otimes v_2 + v_1 \otimes v_1 + v_3 \otimes v_3$,

$\pi(123)(v_1 \otimes v_1 + v_2 \otimes v_2 + v_3 \otimes v_3) = v_3 \otimes v_3 + v_1 \otimes v_1 + v_2 \otimes v_2$.

(c) $\pi(23)(v_1 \otimes v_1 - v_2 \otimes v_2) = v_2 \otimes v_2 - v_1 \otimes v_1$,

$\pi(123)(v_1 \otimes v_1 - v_2 \otimes v_2) = v_3 \otimes v_3 - v_1 \otimes v_1$.

$\pi(23)(v_2 \otimes v_2 - v_3 \otimes v_3) = v_1 \otimes v_1 - v_3 \otimes v_3$,

$\pi(123)(v_2 \otimes v_2 - v_3 \otimes v_3) = v_1 \otimes v_1 - v_2 \otimes v_2.$

We leave it to the reader to show irreducibility.

(d) First note that $\langle v_i, v_i \rangle = 2$, $\langle v_1, v_2 \rangle = 1$, $\langle v_1, v_3 \rangle = -1$, and $\langle v_2, v_3 \rangle = 1$. Then

$$\langle v_1 \otimes v_2 - v_2 \otimes v_1, \sum v_i \otimes v_i \rangle = 2 + 2 + 1 - 2 - 2 - 1 = 0,$$
$$\langle v_1 \otimes v_2 - v_2 \otimes v_1, v_2 \otimes v_2 - v_3 \otimes v_3 \rangle = 2 - 2 - 1 + 1 = 0,$$
$$\langle v_1 \otimes v_2 - v_2 \otimes v_1, v_1 \otimes v_1 - v_2 \otimes v_2 \rangle = 2 - 2 - 2 + 2 = 0,$$
$$\langle \sum v_i \otimes v_i, v_2 \otimes v_2 - v_3 \otimes v_3 \rangle = 1 + 4 + 1 - 1 - 1 - 4 = 0,$$
$$\langle \sum v_i \otimes v_i, v_1 \otimes v_1 - v_2 \otimes v_2 \rangle = 4 + 1 + 1 - 1 - 4 - 1 = 0,$$

(e) $$[\pi(23)]_B = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [\pi(123)]_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

9. We check only the first two, leaving the remainder to the reader.

$$\begin{aligned} w_1 = v_1 \otimes v_2 - v_2 \otimes v_1 &= (e_1 - e_2) \otimes (e_1 - e_3) - (e_1 - e_3) \otimes (e_1 - e_2) \\ &= e_1 \otimes e_1 - e_1 \otimes e_3 - e_2 \otimes e_1 + e_2 \otimes e_3 \\ &\quad -e_1 \otimes e_1 + e_1 \otimes e_2 + e_3 \otimes e_1 - e_3 \otimes e_2 \\ &= (e_3 \otimes e_1 - e_1 \otimes e_3) + (e_1 \otimes e_2 - e_2 \otimes e_1) + (e_2 \otimes e_3 - e_3 \otimes e_2) \end{aligned}$$

One sees that a permutation σ of indices corresponds to change by $sgn(\sigma)$.

$$\begin{aligned} w_2 = v_1 \otimes v_1 + v_2 \otimes v_2 + v_3 \otimes v_3 &= (e_1 - e_2) \otimes (e_1 - e_2) + (e_1 - e_3) \otimes (e_1 - e_3) \\ &\quad +(e_2 - e_3) \otimes (e_2 - e_3) \\ &= 2(e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3) \\ &\quad -(e_1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes e_1 + e_3 \otimes e_1 \\ &\qquad +e_2 \otimes e_3 + e_3 \otimes e_2) \end{aligned}$$

One sees that a permutation σ of indices causes no change, thus a trivial action.

To show orthogonality, we note that only matching monomials contribute in the inner product. Thus the first term in w_2 contributes 0, and the remaining non-vanishing terms cancel in pairs in the inner product.

10. We show that each $\chi_i \otimes \pi$ is the same as applying an inner automorphism to π, so equivalent by Problem Set 3, Problem 10(a). Referring to Solution Set 1, Problem 10(b), consider χ_2. We multiply $\pi(c)$ and $\pi(r)$ by $-I = \pi(r^2)$, so

$$\sigma \otimes \pi(c) = \pi(cr^2), \qquad \sigma \otimes \pi(r) = \pi(r^3).$$

These changes are also implemented by the inner automorphism $\sigma_{cr}(x) = crx(cr)^{-1}$.

For χ_1,

$$\sigma \otimes \pi(r) = \pi(r), \qquad \sigma \otimes \pi(c) = \pi(cr^2),$$

which is implemented by the inner automorphism $\sigma_r(x) = rxr^{-1}$.

One checks directly that $\sigma_3 \otimes \pi$ is implemented by the inner automorphism $\sigma_c = \sigma_{cr^2}$. Alternatively, one notes that $\chi_2\chi_1 = \chi_3$ corresponds to the inner automorphism $\sigma_{cr}\sigma_r = \sigma_{cr^2}$.

For the equivalence between π and π^*, unitary means $\pi^* \cong \overline{\pi}$. Recall that $\overline{\mathbb{C}^2}$ is the vector space $\mathbb{C}^2$ with scalar multiplication $c(x, y) = (\bar{c}x, \bar{c}y)$. One sees immediately that $L : \mathbb{C}^2 \to \overline{\mathbb{C}^2}$ defined by $Lv = \overline{v}$ is an equivalence between $(\pi, \mathbb{C}^2)$ and $(\overline{\pi}, \overline{\mathbb{C}^2})$. Note that each $\pi(g)$ is a real matrix.