INTRO TO GROUP REPS - JULY 9, 2012
PROBLEM SET 5
RT5. SCHUR'S LEMMA

1. (a) Subspace: if $\pi(g)v = v$ and $\pi(g)w = w$ for all g in G, then $\pi(g)(v + w) = v + w$. Likewise $\pi(g)(cv) = c\pi(g)v$. Subrepresentation: if v is in V^G then $\pi(g)v = v \in V^G$ for all g in G.

(b) If L is in $(Hom_{\mathbb{C}}(V, W))^G$ then $\sigma(g)L = \pi'(g)L\pi(g)^{-1} = L$ for all g in G. Thus $\pi'(g)L = L\pi(g)$ for all g in G, and L is an intertwining operator.

2. (a) Note that DA is the matrix A with rescaled rows and AD is the matrix with rescaled columns. Since D has distinct diagonal entries, the off-diagonal entries of A equal zero.

(b) Since the identity transformation is an intertwining operator, the dimension of $Hom_G(V, V)$ is at least one. If (π, W) is a subrepresentation with $W \neq 0, V$, then the orthogonal projection $P_W : V \to W$ is an intertwining operator and not a multiple of the identity. That is, if V is reducible, then the dimension of $Hom_G(V, V)$ is 2 or greater.

(c) Using parts (a) and (b), we show any matrix A that commutes with each $\pi(g)$ is scalar. Each representation has some $\pi(g)$ with distinct diagonal entries ($\pi(c)$ or $\pi(j)$), so A is diagonal by (a). Checking against another generator in each case ($\pi(r)$ or $\pi(i)$) shows that A is a scalar.

3. (a) By observation, the identity matrix I and the matrix M_1 with all entries $= 1$ commute with each $\pi(g)$. In the latter case, each $\pi(g)A = A = A\pi(g)$. To see that all others are linear combinations of I and M_1, consider $A = \pi(g)A\pi(g)^{-1}$ in terms of change of bases. Since $\pi(g)$ merely permutes the standard basis vectors, we have that $a_{i,j} = a_{\pi(g)i,\pi(g)j}$ for all g in S_3. That is, all the diagonal entries of A are equal, and all the off-diagonal entries are equal.

This result is consistent with the decomposition into irreducibles. The orthogonal projections onto the irreducible subspaces are intertwining operators. On the span of $(1, 1, 1)$, $P_W = \frac{1}{3}M_1$, and $P_{W^\perp} = I - P_W$. For instance, since $W^\perp$ consists of all vectors v whose coordinates sum to zero, $P_W v = \frac{1}{3}M_1 v = 0$ and $P_{W^\perp} v = v$. One should verify the intertwining property directly.

(b) Same arguments as part (a). Each $\pi(g)$ is a permutation matrix. Now the projections are $P_W = \frac{1}{n}M_1$ and $P_{W^\perp} = I - P_W$.

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4. (a) If z is in $Z(G)$ then $gz = zg$ for all g in G, so $\pi(z)\pi(g) = \pi(g)\pi(z)$. That is, $\pi(z) : V \to V$ is an intertwining operator, and $\pi(z) = cI$ by Schur's Lemma.

(b) In each case, we have $\pi(z) = -I$.

5. (a) By definition, $Span_G(S)$ is a subspace of V. If v is in $Span_G(S)$ then

$$v = \sum_{g \in G} \sum_{s \in S} c_{g,s}\, \pi(g)s$$

as a sum with only finitely many terms. If g' is in G, then

$$\pi(g')v = \sum_{g} \sum_{s} c_{g,s}\, \pi(g'g)s$$

is in $Span_G(S)$.

(b) If π is irreducible and v is nonzero then $(\pi, Span_G(v))$ is a non-zero subrepresentation by (a), so $Span_G(V) = V$.

Suppose $Span_G(v) = V$ for all nonzero v in V. If (π, W) is a nonzero subrepresentation, then there exists a nonzero v in W, $Span_G(v) = V \subset W$ and $V = W$.

(c) Choose nonzero v in V. Then $V = Span_G(v)$. That is, every w in V can be written in the form

$$w = \sum_{g \in G} c_g \pi(g)v.$$

Thus $S = \{\pi(g)v\}$ contains a basis of V, and V is finite dimensional with dimension $\leq |G|$.

(d) Generates: e_1. Since S_3 acts by permuting coordinates, e_2 and e_3 are in $Span_{S_3}(e_1)$, so $Span_{S_3}(e_1) = \mathbb{C}^3$. Note that π is reducible, but can be generated by a single vector.

Fails to generate: $v = (1, 1, 1)$. Since S_3 permutes coordinates, $Span_{S_3}(v) = \mathbb{C}v$.

6. We label $V^m = V_1 \oplus \dots V_m$ and $V^n = V_{1'} \oplus \cdots \oplus V_{n'}$. Let $I_{i,j'} : V_i \to V_{j'}$ be the identity map for each i, j. Up to a scalar, $I_{i,j'}$ is the only intertwining operator between V_i and $V_{j'}$. We show that $\{I_{i,j'}\}$ is a basis for $Hom_G(V^m, V^n)$ after extending the domain to V^m by zero.

Linear independence: Suppose $S = \sum_{i,j'} c_{i,j'} I_{i,j'} = 0$. Then for all v in V, $S(v, 0, \dots, 0) = (0, \dots, 0)$. Thus each $c_{1,j'} = 0$, and a similar argument shows all $c_{i,j'} = 0$.

Spanning: Define $J_i : V \to V^m$ by $v \mapsto (0, \dots, 0, v, 0, \dots, 0)$ and $P_{j'} : V^n \to V$ by $(0, \dots, 0, v, 0, \dots, 0) \mapsto v$. Suppose S is in $Hom_G(V^m, V^n)$. By Schur's Lemma, $P_{j'} S J_i$ is a scalar multiple of the identity $s_{i,j'} I$, and we see that $S = \sum_{i,j'} s_{i,j'} I_{i,j'}$.

7. (a) The orthogonal complement of a subrepresentation of a unitary representation is always a subrepresentation. If v in $(V^\sigma)^\perp$ generates an irreducible representation of type σ then v is in V^σ. Since we always have that $W \cap W^\perp = \{0\}$, $v = 0$.

Inductively, we can split of the subrepresentations V^σ using orthogonal complements, so that

$$W = V^{\sigma_1} \oplus \cdots \oplus V^{\sigma_k} \quad \text{and} \quad V = W \oplus W^\perp.$$

Since V is finite-dimensional, this process eventually terminates.

(b) This follows immediately from the vanishing part of Schur's Lemma and Problem 6.

8. (a) $c(AB) = ((AB)^{-1})^T = (B^{-1}A^{-1})^T = (A^{-1})^T(B^{-1})^T = c(A)c(B)$. Since the composition of homomorphisms is a homomorphism, $(c \circ \pi, \mathbb{C}^n)$ is a representation also.

(b) If $A = [a_{ij}]$ then $Au_j = \sum_k a_{kj} u_k$. Now

$$A^* u_j^*(u_i) = u_j^*(Au_i) = \sum_k a_{ki} u_j^*(u_k) = a_{ji}$$

or

$$A^* u_j^* = \sum_i a_{ji} u_i^*;$$

that is, with respect to B^*, $[A^*]_{B^*} = [a_{ji}] = [a_{ij}]^T$, the transpose of A. The result immediately applies to dual representations in coordinates.

(c) If π is unitary, then, with respect to an orthonormal basis B,

$$[\pi^*(g)]_{B^*} = [\pi(g^{-1})]_B^T = \overline{[\pi(g)]}_B$$

since each $[\pi(g)]_B$ is unitary matrix.

(d) For a one-dimensional representation, $\chi^*(g) = \chi(g)^{-1}$ in any basis. Since each $|\chi(g)| = 1$, $\chi(g)^{-1} = \overline{\chi(g)}$. For the defining representation $(\pi, \mathbb{C}^2)$ of D_{2n}, each $[\pi(g)]_B$ is orthogonal, so

$$[\pi^*(g)]_{B^*} = [\pi(g^{-1})]_B^T = [\pi(g)]_B = \overline{[\pi(g)]}_B.$$

9. (a) Linear:

$$\begin{aligned}
[\sigma(g)(f+h)](v,w) &= (f+h)(\pi(g)^{-1}v, \pi(g)^{-1}w) \\
&= f(\pi(g)^{-1}v, \pi(g)^{-1}w) + h(\pi(g)^{-1}v, \pi(g)^{-1}w) \\
&= [\sigma(g)f](v,w) + [\sigma(g)h](v,w)
\end{aligned}$$

Group action: as usual,

$$\begin{aligned}
[\sigma(g_1g_2)f](v,w) &= f(\pi(g_1g_2)^{-1}v, \pi(g_1g_2)^{-1}w) \\
&= f(\pi(g_2)^{-1}\pi(g_1)^{-1}v, \pi(g_2)^{-1}\pi(g_1)^{-1}w) \\
&= [\pi(g_2)f](\pi(g_1)^{-1}v, \pi(g_1)^{-1}w) = [\pi(g_1)[\pi(g_2)f]](v,w)
\end{aligned}$$

(b) Linear: suppose c in $\mathbb{C}$ and T_1, T_2 in $Hom_{\mathbb{C}}(V,V)$. Then

$$i(cT_1 + T_2)(v,w) = \langle (cT_1 + T_2)v, w\rangle = c\langle T_1 v, w\rangle + \langle T_2 v, w\rangle = c(i(T_1)(v,w)) + i(T_2)(v,w).$$

Suppose $i(T)(v,w) = 0$ for all v, w in V. If we choose an orthonormal basis $B = \{u_i\}$ for V, then $c_{i,j} = \langle Te_j, e_i\rangle = 0$ for all i, j. That is, with respect to the orthonormal basis, the associated matrix of T is $[T]_B = 0$. So i is one-one. To see onto, we note that the sesquilinear forms, defined on the basis B,

$$f_{i,j}(u_k, u_l) = \begin{cases} 1 & \text{if } i = k, j = l \\ 0 & \text{otherwise} \end{cases}$$

form a basis of $Sesq(V)$.

Intertwining Operator: suppose T is in $Hom_{\mathbb{C}}(V,V)$ and v, w in V. Then

$$\begin{aligned} i(\sigma(g)T)(v,w) &= i(\pi(g)T\pi(g)^{-1})(v,w) = \langle \pi(g)T\pi(g)^{-1}v, w\rangle \\ &= \langle T\pi(g)^{-1}v, \pi(g)^{-1}w\rangle = i(T)(\pi(g)^{-1}v, \pi(g)^{-1}w) \\ &= \sigma(g)(iT)(v,w) \end{aligned}$$

Unitary Equivalence: Define

$$\langle f_1, f_2\rangle_S = \sum_{i,j} f_1(u_i, u_j)\overline{f_2(u_i, u_j)}.$$

Then

$$\begin{aligned} \langle i(T_1), i(T_2)\rangle_S &= \sum_{i,j} \langle T_1u_i, u_j\rangle \overline{\langle T_2u_i, u_j\rangle} \\ &= \sum_{i,j} \langle T_1u_i, u_j\rangle\langle u_j, T_2u_i\rangle = \sum_i \langle \langle \sum_j T_1u_i, u_j\rangle u_j, T_2u_i\rangle \\ &= \sum_i \langle T_1u_i, T_2u_i\rangle = \sum_i \langle T_2^*T_1u_i, u_i\rangle \\ &= Trace(T_2^*T_1) = Trace(T_1T_2^*). \end{aligned}$$

This also shows that the definition of $\langle\cdot,\cdot\rangle_S$ is independent of the choice of orthonormal basis and in turn that the representation on $Sesq(V)$ is unitary.

(c) Since $X = X^*$,

$$\begin{aligned} i(X)(v,w) &= \langle Xv, w\rangle = \langle v, X^*w\rangle \\ &= \langle v, Xw\rangle = \overline{\langle Xw, v\rangle} \\ &= \overline{i(X)(w,v)} \end{aligned}$$

So $i(X)$ is a Hermitian form.

(d) If $\sigma(g)f = f$ for all g in G, then $f(\pi(g)^{-1}v, \pi(g)^{-1}w) = f(v,w)$. That is, $f = i(T)$ where T is in $Hom_G(V,V)$. To see an example of such a form, let (π, W) be a subrepresentation of (π, V) and $P_W : V \to W$ the corresponding orthogonal projection on W. Then $i(P_W)$ is such a form. If π is irreducible, then the only intertwining operators are scalar multiples of the identity, so f is a multiple of $\langle\cdot,\cdot\rangle$.

10. (a) Suppose $A = [a_{ij}]$ is a unitary matrix. Then $AA^* = I$, or

$$\sum_j a_{ij}\overline{a_{kj}} = \begin{cases} 1 & i = k, \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$\begin{aligned} \sum_j \langle u, Av_j\rangle Av_j &= \sum_j \langle u, \sum_i a_{ij}v_i\rangle \sum_k a_{kj}v_k \\ &= \sum_{i,j,k} \overline{a_{ij}}a_{kj}\langle u, v_i\rangle v_k \\ &= \sum_k \langle u, v_k\rangle v_k. \end{aligned}$$

(b) Let $B = \{v_i\}$ and $C = \{u_j\}$. The j-th column of $[T]_{B,C}$ is the coordinate vector $[Tv_j]_C$. Since $Tv_j = \sum_i \langle v_j, u_i\rangle u_i$, the (i, j)-th entry of $[T]_{B,C}$ equals $\langle v_j, u_i\rangle$.

(c) Suppose $\langle\cdot,\cdot\rangle$ in part (a) is invariant. Then

$$\begin{aligned} T\pi(g)u &= \sum_j \langle \pi(g)u, v_j\rangle v_j = \sum_j \langle u, \pi'(g)^{-1}v_j\rangle v_j \\ &= \sum_j \langle u, v_j\rangle \pi'(g)v_j = \pi'(g)Tu \end{aligned}$$

since $\pi'(g)$ is unitary and T is independent of orthonormal basis chosen by part (a). Thus T is in $Hom_G(V, V') = 0$, and each $\langle u_i, v_j\rangle = 0$ since $\{v_j\}$ is a basis for V'.