INTRO TO GROUP REPS - JULY 16, 2012
SOLUTION SET 6
RT6. FUNCTION SPACES

1. (a) Since $\pi(g)e_x = e_{\sigma(g)x}$ is another basis vector, $\pi(g)$ permutes the basis vectors; that is, $[\pi(g)]_B$ is just the identity matrix with its columns permuted. Thus there is exactly a single 1 in each row and column.

(b) If g is in G, then

$$\pi(g)v = \sum_{x\in X} c_x e_{\sigma(g)x} = \sum_{\sigma(g^{-1})y\in X} c_{\sigma(g^{-1})y} e_y = \sum_{y\in X} c_{\sigma(g^{-1})y} e_y$$

by substituting $y = \sigma(g)x$ and then noting that $\sigma(g^{-1})$ is a bijection on X. Thus

$$\langle v, w\rangle = \sum_{x\in X} c_x \overline{d_x} = \sum_{y\in X} c_{\sigma(g^{-1})y}\, \overline{d_{\sigma(g^{-1})y}} = \langle \pi(g)v, \pi(g)w\rangle.$$

(c) Let O_x be an orbit in X by G. Then $O_x = \{\sigma(g)x \mid g \in G\}$. Consider the span of

$$v_x = \sum_{x\in O_x} e_x.$$

Since each $\pi(g)$ is a bijection on O_x, $\pi(g)v_x = v_x$ for all g in G, so trivial.

Suppose $w = \sum_{x\in X} c_x e_x$ is a nonzero vector with trivial action by π, and let

$$w_y = \sum_{x\in O_y} c_y e_y.$$

Then

$$\pi(g)w_y = w_{\sigma(g)y} = w_y$$

for all g in G. By transitivity of the group action on O_y, each $c_y = c_{\sigma(g)y}$. So $w_y = c_y v_y$, and $w = \sum c_y v_y$.

(d) Define $A : V^* \to L^2(X)$ by $A(e_x^*) = \delta_x$ where $\{e_x^*\}$ is the dual basis to $\{e_x\}$, and

$$\delta_x(x') = \begin{cases} 1 & x = x' \\ 0 & \text{otherwise} \end{cases}.$$

One sees immediately that $\{\delta_x\}$ is an orthonormal basis for $L^2(X)$. Then extend A linearly.

Vector space isomorphism: A is linear by definition. If $0 = A(\sum c_x e_x) = \sum c_x \delta_x$, then evaluation at each g shows that each $c_g = 0$, so the kernel of A is zero. Since $\{e_g\}$ and $\{\delta_g\}$ are bases of V^* and $L^2(G)$ respectively, onto follows immediately.

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Intertwining Operator: Since

$$[\pi^*(g)e_x^*](e_{x'}) = e_x^*(\pi(g^{-1})e_{x'}) = e_x^*(e_{\sigma(g^{-1})x'}),$$

we have that $\pi^*(g)e_x^* = e_{\sigma(g)x}^*$. Now for all g in G,

$$A(\pi^*(g)e_x^*) = A(e_{\sigma(g)x}) = \delta_{\sigma(g)x}.$$

On the other hand,

$$L(g)[Ae_x^*] = L(g)\delta_x = \delta_{\sigma(g)x}$$

by a similar argument.

(e) From part (a), each $\pi(g)$ may be represented as a permutation matrix. Thus the trace is the number of 1s on the diagonal. This occurs when $\pi(g)e_x = e_x$ or $\sigma(g)x = x$.

2. Let $G = X = \{\pm 1\}$ and let $L^2(G)$ be the vector space of functions from G to $\mathbb{C}$. Let G act on $L^2(G)$ by left translation: $[L(g)f](x) = f(g^{-1}x)$, and we define an invariant Hermitian inner product on $L^2(G)$ by

$$\langle f, h\rangle = \frac{1}{2}[f(1)\overline{h(1)} + f(-1)\overline{h(-1)}].$$

(a) Linear: $L(g)(cf+h)(x) = (cf+h)(g^{-1}x) = c(f(g^{-1}x)) + h(g^{-1}x) = c((L(g)f)(x)) + (L(g)(h))(x)$.

The inner product is clearly invariant under $L(1)$:

$$\begin{aligned}
\langle L(-1)f, L(-1)h\rangle &= \frac{1}{2}[L(-1)f(1)\overline{L(-1)h(1)} + L(-1)f(-1)\overline{L(-1)h(-1)}] \\
&= \frac{1}{2}[f(-1)\overline{h(-1)} + f(1)\overline{h(1)}] \\
&= \langle f, h\rangle.
\end{aligned}$$

$$\begin{aligned}
\text{(b) } \langle \chi_{triv}, \chi_{triv}\rangle &= \frac{1}{2}[1\cdot 1 + 1\cdot 1] = 1, \\
\langle \chi_{sgn}, \chi_{sgn}\rangle &= \frac{1}{2}[1\cdot 1 + (-1)\cdot(-1)] = 1, \\
\langle \chi_{triv}, \chi_{sgn}\rangle &= \frac{1}{2}[1\cdot 1 + (1)\cdot(-1)] = 0.
\end{aligned}$$

(c)

$$\begin{aligned}
(P_{triv}P_{triv}f)(x) &= \frac{1}{2}[P_{triv}f(x) + P_{triv}f(-x)] \\
&= \frac{1}{2}[\frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(-x) + f(x))] \\
&= \frac{1}{2}[f(x) + f(-x)] = P_{triv}f(x)
\end{aligned}$$

$$\begin{aligned}
\langle P_{triv}f, h\rangle &= \frac{1}{2}[P_{triv}f(1)\overline{h(1)} + P_{triv}f(-1)\overline{h(-1)}] \\
&= \frac{1}{2}[\frac{1}{2}(f(1)+f(-1))\overline{h(1)} + \frac{1}{2}(f(-1)+f(1))\overline{h(-1)}] \\
&= \frac{1}{2}[\frac{1}{2}(f(1)\overline{P_{triv}h(1)} + \frac{1}{2}f(-1)\overline{P_{triv}h(-1)}] \\
&= \langle f, P_{triv}h\rangle
\end{aligned}$$

$$\begin{aligned}
(P_{sgn}P_{sgn}f)(x) &= \frac{1}{2}[P_{sgn}f(x) - P_{sgn}f(-x)] \\
&= \frac{1}{2}[\frac{1}{2}(f(x)-f(-x)) - \frac{1}{2}(f(-x)-f(x))] \\
&= \frac{1}{2}[f(x)-f(-x)] = P_{sgn}f(x)
\end{aligned}$$

$$\begin{aligned}
\langle P_{sgn}f, h\rangle &= \frac{1}{2}[P_{sgn}f(1)\overline{h(1)} + P_{sgn}f(-1)\overline{h(-1)}] \\
&= \frac{1}{2}[\frac{1}{2}(f(1)-f(-1))\overline{h(1)} + \frac{1}{2}(f(-1)-f(1))\overline{h(-1)}] \\
&= \frac{1}{2}[\frac{1}{2}(f(1)\overline{P_{sgn}h(1)} + \frac{1}{2}f(-1)\overline{P_{sgn}h(-1)}] \\
&= \langle f, P_{sgn}h\rangle
\end{aligned}$$

$$\begin{aligned}
\text{(d) } P_{triv}f(1) &= \frac{1}{2}(3+(-4)) = -\frac{1}{2}, \quad P_{triv}f(-1) = \frac{1}{2}(-4+(3)) = -\frac{1}{2}, \\
P_{sgn}f(-1) &= \frac{1}{2}(3-(-4)) = \frac{7}{2}, \quad P_{sgn}f(-1) = \frac{1}{2}(-4-(3)) = -\frac{7}{2}.
\end{aligned}$$

Verfiying Fourier's Trick, $c_{triv} = \langle f, \chi_{triv}\rangle = -\frac{1}{2}$ and $c_{sgn} = \langle f, \chi_{sgn}\rangle = \frac{7}{2}$.

Verifying Parseval's Identity, $\frac{1}{2}(3^2 + (-4)^2) = \frac{25}{2} = (-\frac{1}{2})^2 + (\frac{7}{2})^2$.

3. Let $(\pi, (\mathbb{C}^4)$ be the permutation representation of D_8. With respect to the basis $B = \{e_i\}$,

$$[\pi(1234)]_B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad [\pi(14)(23)]_B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Since these matrices are orthogonal, $[\pi^*(g)]_{B^*} = [\pi(g)]_B^T$ for all g in G.

First note that the constant functions $W_1 = Span(e_1^* + e_2^* + e_3^* + e_4^*)$ yield a trivial subrepresentation.

The orthogonal complement to W_1 consists of all vectors $\sum_i c_i e_i^*$ with $\sum_i c_i = 0$. We look for joint eigenvectors of $\pi^*(r)$ and $\pi^*(c)$. Consider the function $w_2 = e_1^* - e_2^* + e_3^* - e_4^*$. Both $\pi^*(c)$and $\pi^*(r)$ act by -1. The span of this eigenvector is a subrepresentation of type χ_2 (SS1) for D_8.

The orthogonal complement to W_1 and W_2 consists of all $ae_1^* + be_2^* + ce_3^* + de_4^*$ with $a = -c$ and $b = -d$. Diagonalizing $\pi(c)$ on this space gives eigenvectors $e_1^* + e_2^* - e_3^* - e_4^*$ and $e_1^* - e_2^* - e_3^* + e_4^*$ with eigenvalues -1 and 1. Since neither is an eigenvector for $\pi(r)$, the two dimensional span yields the irreducible subrepresentation W_3 of D_8.

4. (a) Mostly straightforward. We show that $\langle f, f\rangle \geq 0$ and $= 0$ if and only if $f = 0$. Note that $|f(t)|^2$ is real valued and non-negative. Since f is piecewise continuous on $[0, 2\pi)$, so is $|f(x)|^2$, and the definite integral is just the area between the graph and the x-axis. Thus $\langle f, f\rangle \geq 0$. If $|f|^2 = 0$, then the area is zero, and, since continuous, $|f(x)|^2 = 0$ except possibly at the finitely-many discontinuities. Thus $f(x) = 0$ everywhere except at finitely many points.

For t' in $\mathbb{R}$, periodicity and change of variables imply unitarity:

$$\langle L(t')f, L(t')h\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t-t')\overline{h(t-t')}\ dt = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{h(t)}\ dt = \langle f, h\rangle.$$

Of course, we check that L is linear:

$$(L(t')(cf + h))(t) = (cf + h)(t - t') = c(f(t - t')) + h(t - t') = c[L(t')f](t) + [L(t')h](t).$$

(b) Fix t in $\mathbb{R}$.

$$L(t')F_n(t) = F_n(t - t') = e^{in(t-t')} = e^{-int'}e^{int} = \chi_{-n}(e^{it'})F_n(t) = \overline{\chi_n(e^{it'})}F_n(t),$$

and

$$R(t')F_n(t) = F_n(t + t') = e^{in(t+t')} = e^{int'}e^{int} = \chi_n(e^{it'})F_n(t).$$

(c) First note that

$$\frac{1}{2\pi}\int_0^{2\pi} 1\ dt = 1$$

and, when $n \neq 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{int}\ dt = \frac{1}{2\pi}\int_0^{2\pi} cos(nt) + isin(nt)\ dt = 0.$$

Thus

$$\langle F_m, F_n\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)t}\ dt = \begin{cases} 1 & m = n \\ 0 & \text{otherwise} \end{cases}.$$

5. (a) Since $c_n = \langle F, F_n \rangle$, we have

$$\begin{aligned}\langle F, F_n \rangle &= \frac{1}{2\pi} \int_0^{2\pi} F(t) e^{-int}\, dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{0} -e^{-int}\, dt + \frac{1}{2\pi} \int_0^{\pi} e^{-int}\, dt \\ &= \frac{1}{2n\pi i} [(1-(-1)^n) - ((-1)^n - 1)] = \frac{1}{n\pi i} (1-(-1)^n)\end{aligned}$$

Note that $c_n = 0$ when n is even, and, since F is real valued, $c_{-n} = \overline{c_n} = -c_n$

Thus

$$F(t) = \sum_{n>0\ odd} \frac{2}{n\pi i} (e^{int} - e^{-int}) = \sum_{n>0\ odd} \frac{4}{n\pi}\ sin(nt)$$

(b) First $\langle F, F \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(t)|^2 dt = 1$. But Parseval's Identity states that this length squared equals

$$\sum_i |c_n|^2 = 2 \sum_{n>0\ odd} \frac{4}{n^2 \pi^2}.$$

Simplifying gives that

$$\sum_{n>0\ odd} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

(c) Note that

$$\begin{aligned}\sum_{n>0} \frac{1}{n^2} &= \sum_{n>0\ odd} \frac{1}{n^2} + \sum_{n>0\ even} \frac{1}{n^2} \\ &= \frac{\pi^2}{8} + \frac{1}{4} \sum_{n>0} \frac{1}{n^2}\end{aligned}$$

This gives

$$\sum_{n>0} \frac{1}{n^2} = \frac{4}{3} \frac{\pi^2}{8} = \frac{\pi^2}{6}.$$

Calculating the first n terms,

n	$\sum$
10	1.549768
100	1.634984
500	1.642936
750	1.643602
$\pi^2/6$	1.644934

6. (a) Induction on the size of A and B: Base case: when $n = 1$, $Trace(AB) = a_{11} b_{11} = Trace(BA)$.

Induction step: Suppose true when size is n. Suppose A and B have size $n+1$, and consider the products

$$\begin{bmatrix} A' & a_{*n+1} \\ a_{n+1*} & a_{n+1,n+1} \end{bmatrix} \begin{bmatrix} B' & b_{*n+1} \\ b_{n+1*} & b_{n+1,n+1} \end{bmatrix} \qquad \begin{bmatrix} B' & b_{*n+1} \\ b_{n+1*} & b_{n+1,n+1} \end{bmatrix} \begin{bmatrix} A' & a_{*n+1} \\ a_{n+1*} & a_{n+1,n+1} \end{bmatrix}$$

Here we bookkeep for the diagonal entries only: in the AB product, we have upper left blocks plus last column of A dot last row of B plus last row of A dot last column of B.

$$Trace(A'B') + \sum_i a_{i,n+1}b_{n+1,i} + \sum_j a_{n+1,j}b_{j,n+1}.$$

In the other direction,

$$Trace(B'A') + \sum_j b_{j,n+1}a_{n+1,j} + \sum_i b_{n+1,i}a_{i,n+1}.$$

Applying the induction hypothesis to A' and B', the result follows.

(b)

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix},$$

in which case

$$8 = Trace(ABC) \neq Trace(ACB) = 7.$$

7. (a) Since $Trace(AB) = Trace(BA)$,

$$Trace(PAP^{-1}) = Trace(P^{-1}PA) = Trace(A).$$

Let $T : V \to V$ be a linear transformation. Suppose we change from basis B to basis C. Then

$$Trace([T]_B) = Trace(P_{B\leftarrow C}[T]_C P_{B\leftarrow C}^{-1}) = Trace([T]_C).$$

We verify that the definitions agree. If $Tu_i = \sum_j A_{ji}u_j$ then $u_i^*(Tu_i) = A_{ii}$ and

$$Trace(T) = \sum_i u_i^*(Tu_i) = \sum_i A_{ii}.$$

When V is an inner product space, the formula follows by noting that the dual of an orthonormal basis $\{u_i\}$ is $\{\langle \cdot, u_i \rangle\}$.

(b) Suppose $PAP^{-1} = D$, where D is a diagonal matrix. By (a),

$$Trace(A) = Trace(PAP^{-1}) = Trace(D).$$

The result follows since the diagonal entries of D are the eigenvalues of D, and A and D have the same eigenvalues with multiplicities.

(c): Induction proof on the size of A: Base case: $n = 1$: $p_A(x) = (x - a_{11})$.

Induction step: Since we work over $\mathbb{C}$, there exists at least one eigenvalue λ for A and at least one nonzero eigenvector v. If we complete $\{v\}$ to any basis of $\mathbb{C}^{n+1}$, then

$$[A]_B = \begin{bmatrix} \lambda & * \\ 0 & A' \end{bmatrix}.$$

By the induction hypothesis,

$$\begin{aligned} p_A(x) &= (x-\lambda)p_{A'}(x) \\ &= (x-\lambda)(x^n - Trace(A')x^{n-1} + \dots) \\ &= x^{n+1} - (\lambda + Tr(A'))x^n + \dots. \end{aligned}$$

Thus the induction step holds, and the result follows.

(d) One can adapt the proof of part (c), or just note that

$$p_A(x) = (x-\lambda_1)\dots(x-\lambda_n) = x^n - (\sum_i \lambda_i)x^{n-1} + \dots.$$

8. (a) $\chi_\pi(e) = Trace(\pi(e)) = Trace(I) = dim(V)$.

(b) If $L : V \to V'$ is an equivalence, then $\pi'(g)L = L\pi(g)$ or $\pi(g) = L^{-1}\pi'(g)L$, and

$$\chi_\pi(g) = Trace(\pi(g)) = Trace(L^{-1}\pi'(g)L) = Trace(\pi'(g)) = \chi_{\pi'}(g).$$

A similar argument gives

$$\chi_\pi(gxg^{-1}) = Trace(\pi(gxg^{-1})) = Trace(\pi(g)\pi(x)\pi(g)^{-1}) = Trace(\pi(x)) = \chi_\pi(x).$$

(c) Each $\pi(g)$ is diagonalizable with unit eigenvalues. If $Av = \lambda v$ with A invertible, then $A^{-1}v = \lambda^{-1}v$. Thus A and A^{-1} have the same eigenvectors, but the eigenvalues are inverted. Since λ is unit, $\lambda^{-1} = \overline{\lambda}$. Now

$$\chi_\pi(g^{-1}) = Trace(\pi(g^{-1})) = Trace(\pi(g)^{-1}) = \sum_i \lambda^{-1} = \sum_i \overline{\lambda} = \overline{\chi_\pi(g)}.$$

(d) If we choose bases $B = \{u_i\}$ and $C = \{v_i\}$ for V and V', then $[\pi \oplus \pi'(g)]_{B\cup C}$ is block diagonal with blocks $[\pi(g)]_B$ and $[\pi'(g)]_C$. Thus

$$\chi_{\pi\oplus\pi'}(g) = Trace((\pi \oplus \pi')(g)) = Trace(\pi(g)) + Trace(\pi'(g)) = \chi_\pi(g) + \chi_{\pi'}(g).$$

Next, with respect to the dual basis B^*, $[\pi^*(g)]_{B^*} = [\pi(g)^{-1}]_B^T$. Thus

$$\chi_{\pi^*}(g) = Trace(\pi^*(g)) = Trace([\pi(g)^{-1}]_B^T) = Trace([\pi(g)^{-1}]_B) = \overline{\chi_\pi(g)}.$$

Let B^* and C^* be the dual bases to B and C. Then $B \otimes C = \{u_i \otimes v_j\}$ has dual basis $B^* \otimes C^* = \{u_i^* \otimes v_j^*\}$ with

$$(v^* \otimes w^*)(v' \otimes w') = v^*(v')w^*(w').$$

Now

$$\begin{aligned}
\chi_{\pi\otimes\pi'}(g) &= Trace((\pi \otimes \pi')(g)) = \sum_{i,j}(u_i^* \otimes v_j^*)[(\pi \otimes \pi')(g)(u_i \otimes v_j)]. \\
&= \sum_{i,j} u_i^*(\pi(g)u_i)\ v_j^*(\pi'(g)v_j) \\
&= [\sum_i u_i^*(\pi(g)u_i)]\ [\sum_j v_j^*(\pi'(g)v_j)] \\
&= Trace(\pi(g))\ Trace(\pi'(g)) = \chi_\pi(g)\chi_{\pi'}(g)
\end{aligned}$$

9. (a) We have four classes: $\{e\}, \{(ij)(kl)\}, \{(123), (134), (243), (142)\}$, and

$$\{(132), (143), (234), (124)\}.$$

By the sum of squares formula,

$$12 = 1^2 + 1^2 + 1^2 + x^2.$$

Because there are only three inequivalent (one-dimensional) characters, there is only one more irreducible class with dimension 3.

(b) e: the representation is three dimensional, so $\chi_\pi(e) = 3$.

$(123), (132)$: because trace is independent of basis chosen, we can orient the tetrahedron as we wish. In either case, we rotate by $\frac{2\pi}{3}$ clockwise or counter-clockwise, and in some basis

$$[\pi(g)]_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\pm 2\pi i/3) & -sin(\pm 2\pi i/3) \\ 0 & sin(\pm 2\pi i/3) & cos(\pm 2\pi i/3) \end{bmatrix}.$$

In either case, the eigenvalues are

$$1, \quad e^{\pm 2\pi i/3} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}.$$

Thus $\chi_\pi(123) = \chi_\pi(132) = 0$.

$(12)(34)$: If we rotate about the segment joining the midpoints of 12 and 34 by π, any plane parallel to 12 and 34 is rotated by π. Thus the eigenvalues are $1, -1, -1$ and $\chi_\pi((12)(34)) = -1$.

(c) With respect to the standard basis of $\mathbb{R}^3$, we see by observation that

$$[\pi((12)(34))]_B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For $g = (123)$, we solve the equations

$$Tv_1 = v_2, \quad Tv_2 = v_3, \quad Tv_3 = v_1$$

to find each Te_i. Considering each $T(v_i + v_j)$ yields

$$[\pi(123)]_B = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Taking traces, we verify part (b).

(d) Since proper subrepresentations must have dimension one or two, we must have a joint eigenvector v for all $\pi(g)$ if reducible. Consideration of $\pi(g)v$ for $g = (12)(34), (14)(23)$, and $(13)(24)$ shows that v is a multiple of a standard basis vector. By applying $\pi(123)$ to v, we see that $v = 0$.

Alternatively, we use the converse of Schur's Lemma. If A commutes with all $\pi(g)$, then consideration of $g = (12)(34), (13)(24), (14)(23)$ shows that A is diagonal. The only diagonal matrices that commute with $\pi(123)$ are scalar multiples of the identity. Thus π is irreducible.

(e) The number to the right of the class representative is the number of elements in the class. When we compute inner products of rows, we weight by these numbers. For instance, the inner product of rows χ_0 and π is

$$\frac{1}{12}(1 \cdot \overline{3} + 4 \cdot 1 \cdot \overline{0} + 4 \cdot 1 \cdot \overline{0} + 3 \cdot 1 \cdot \overline{-1}) = 0.$$

A_4	e (1)	(123) (4)	(132) (4)	(12)(34) (3)
χ_0	1	1	1	1
χ_1	1	ω	ω^2	1
χ_2	1	ω^2	ω	1
π	3	0	0	-1

One verifies orthonormality directly, noting that $1 + \omega + \omega^2 = 0$ if $\omega = e^{2\pi i/3}$.

10. (a) Since $f(gxg^{-1}) = f(x)$ for all x and g, we can replace x with xg to get

$$f(g(xg)g^{-1}) = f(xg) \quad \text{or} \quad f(gx) = f(xg).$$

$$\begin{aligned} (b)\ (Af)(hxh^{-1}) &= \frac{1}{|G|}\sum_g f(ghxh^{-1}g^{-1}) \\ &= \frac{1}{|G|}\sum_g f(ghx(gh)^{-1}) \\ &= \frac{1}{|G|}\sum_{g'} f(g'xg'^{-1}) = (Af)(x). \end{aligned}$$

(c) Let O_x be the conjugacy class of x in G and define

$$D_x(y) = \begin{cases} 1 & y \in O_x \\ 0 & \text{otherwise} \end{cases}.$$

Then $\{\sqrt{|O_x|}D_x\}$ is an orthonormal basis for $Class(G)$ with respect to the $L^2(G)$-norm.

(d) $\chi_\pi(gxg^{-1}) = Trace(\pi(gxg^{-1})) = Trace(\pi(g)\pi(x)\pi(g)^{-1}) = Trace(\pi(x)) = \chi_\pi(x)$.

(e) When $G = S_3$, there are three conjugacy classes:

$$\{e\},\ \{(12),(23),(13)\},\ \text{and } \{(123),(132)\}.$$

There are also three classes of irreducible representations: χ_{triv}, χ_{sgn}, and the irreducible two-dimensional representation.

When $G = D_8$, there are 5 conjugacy classes: $\{e\}$, $\{r^2\}$, $\{r, r^3\}, \{c, cr^2\}$, and $\{cr, cr^3\}$. There are also five classes of irreducible representations: 4 characters and the irreducible two-dimensional representation.

When $G = Q$, there are also 5 classes: $\{1\}, \{-1\}, \{\pm i\}, \{\pm j\}$, and $\{\pm k\}$. A similar statement holds as with D_8 for irreducible classes.

When $G = A_4$, we have four classes: $\{e\}, \{(ij)(kl)\}, \{(123),(134),(243),(142)\}$, and $\{(132),(143),(234),(124)\}$. There are three classes of irreducible characters and one class for the irreducible three-dimensional representation.

When G is abelian, $|G| = |G^*|$. Each conjugacy class in G is a singleton. Thus the number of classes for irreducible representations equals the number of conjugacy classes in G. This also holds for non-abelian finite groups; to be seen later.