INTRO TO GROUP REPS - JULY 23, 2012
SOLUTION SET 7
RT7. FINITE ABELIAN GROUPS

1. (a) If we consider the character table as a square matrix A, then $AA^* = |G|I$ or $A^{-1} = \frac{1}{|G|}A^*$. Since A^{-1} commutes with A, we have that $A^*A = |G|I$. Thus the rows of A^* are orthogonal, which is equivalent to the orthogonality of the columns of A.

(b) Character of G^*: $e_g(\chi_1\chi_2) = (\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g) = e_g(\chi_1)e_g(\chi_2)$.

Isomorphism: Define $\pi : G \to G^{**}$ by $\pi(g) = e_g$. Then

$$\pi(gh)(\chi) = e_{gh}(\chi) = \chi(gh) = \chi(g)\chi(h) = (e_g e_h)(\chi).$$

Thus $\pi(gh) = \pi(g)\pi(h)$, and π is a homomorphism.

Since $|G| = |G^*| = |G^{**}|$, it is enough to show π is one-one. Suppose $e_g(\chi) = 1$ for all χ in G^*. Then $\chi(g) = 1$ for all χ in G^*. Suppose $g \neq e$. Since $\chi(e) = 1$ for all χ in G^*, this implies that the character table of G has two columns of 1s, and the table is singular as a matrix, a contradiction since characters form an orthonormal basis of $L^2(G)$.

(c) By part (b), the transpose of character table of G is the character table of G^*. Thus the rows of the transpose are orthogonal, but these correspond to the columns for the character table of G.

2. (a) If $\chi(g) = \overline{\chi(g)} = \chi(g)^{-1}$ for all g in G, then $\chi^2 = I$ for all χ in G^*. Since G is isomorphic to G^*, $g^2 = e$ for each g in G. In turn, G must be a finite product of $\mathbb{Z}/2$ groups.

(b) If G is non-abelian, then $G/[G, G]$, the abelianization of G, is a product of $\mathbb{Z}/2$ groups. One should compare D_8 or Q versus A_4.

3. (a) We have that $\omega^n = e^{2\pi i} = 1$, so ω is an n-th root of unity. To see primitive, Euler's formula gives

$$e^{2\pi i/n} = cos(2\pi/n) + sin(2\pi/n)i$$

and

$$(e^{2\pi/n})^k = cos(2k\pi/n) + sin(2k\pi/n)i.$$

If $\omega^k = 1$ then $cos(2k\pi/n) = 1$ or k is a multiple of n. Thus ω is primitive.

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(b) DeMoivre's Theorem implies that the roots of $p(z) = z^n - 1$ are $(e^{2\pi i/n})^k$ for $0 \leq k < n$. These are distinct by Euler's Formula or by noting that $p'(z) = nz^{n-1}$ has no roots in common with $p(z)$. The primitive roots are those roots with k such that $(k, n) = 1$.

(c) If we divide $p(z) = z^n - 1$ by $z - 1$, the quotient is $q(z) = 1 + z + \cdots + z^{n-1}$. This can be seen using synthetic division or computing $(z-1)q(z)$ directly. If ω is an n-th root not equal to 1, then $0 = p(\omega) = (\omega - 1)q(\omega)$ and $q(\omega) = 0$.

(d) Irreducible implies one dimensional. Since $\mathbb{Z}/n$ is cyclic with generator 1, χ is determined by $\chi(1) = \omega$, where ω can be any n-th root of unity. To be faithful, the kernel of χ must contain 1 only. That is, $\chi(\omega^k) = 1$ implies that k is a multiple of n. Thus ω must be a primitive root.

4. (a) Let $\omega = e^{2\pi i/n}$.

$\mathbb{Z}/n$	0	1	$\dots$	$n-1$
χ_0	1	1	$\dots$	1
χ_1	1	ω	$\dots$	ω^{n-1}
χ_2	1	ω^2	$\dots$	$\omega^{2(n-1)}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
χ_{n-1}	1	ω^{n-1}	$\dots$	$\omega^{(n-1)^2}$

(b) Unit Rows: For the k-th row,

$$\langle \chi_k, \chi_k \rangle = \frac{1}{n} \sum_{i=0}^{n-1} \chi_k(i)\overline{\chi_k(i)} = \frac{1}{n} \sum_{i=0}^{n-1} \omega^{ik}\omega^{-ik} = 1.$$

Orthogonal Rows: if $j \neq k$,

$$\langle \chi_j, \chi_k \rangle = \frac{1}{n} \sum_{i=0}^{n-1} \chi_j(i)\overline{\chi_k(i)} = \frac{1}{n} \sum_{i=0}^{n-1} \omega^{ij}\omega^{-ik} = \frac{1}{n} \sum_{i=0}^{n-1} (\omega^{j-k})^i = 0.$$

Orthogonal Columns: if $j \neq k$,

$$\sum_{i=0}^{n-1} \chi_i(j)\overline{\chi_i(k)} = \sum_{i=0}^{n-1} \omega^{ij}\omega^{-ik} = \sum_{i=0}^{n-1} \omega^{i(j-k)} = 0.$$

(c) In general, $c_i = \langle f_1, \chi_i \rangle = \frac{1}{n}\omega^{-i}$, so

$$f_1(k) = \sum_{i=0}^{n-1} c_i \chi_i(k) = \frac{1}{n} \sum_{i=0}^{n-1} \omega^{-i} \chi_i(k),$$

verifying Fourier's Trick.

For Parseval's Identity,

$$\langle f_1, f_1 \rangle = \frac{1}{n} = \sum_{i=0}^{n-1} \frac{1}{n^2} = \sum_{i=0}^{n-1} |c_i|^2.$$

5. (a) Since every element of G has order 2, each $\chi(g) = \pm 1$. Because $\{e_i\}$ is a generating set with the only relations $e_i + e_j = e_j + e_i$, all such choices of $\chi(e_i)$ yield a representation. Thus there are 2^n characters.

(b) Induction on n: Base case: $n = 2$. We have seen that the characters of $\mathbb{Z}/2 \times \mathbb{Z}/2$ are orthogonal in $L^2(G)$.

Suppose the result holds for n factors, and we consider the group with $n + 1$ factors. Suppose $\chi \neq \chi'$. Then every element in G may be written in the form $g + 0$ or $g + e_{n+1}$ where g ranges over elements generated by $\{e_1, \ldots, e_n\}$. Thus, summing g over elements from the first n factors,

$$\sum_g \chi(g)\overline{\chi'(g)} + \sum_g \chi(g + e_{n+1})\overline{\chi'(g + e_{n+1})} = 2\sum_g \chi(g)\overline{\chi'(g)} = 0$$

since χ and χ' are characters when restricted to the group generated by $\{e_1, \ldots, e_n\}$.

(c) Recall

$\mathbb{Z}/2 \times \mathbb{Z}/2$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
χ_0	1	1	1	1
χ_1	1	-1	1	-1
χ_2	1	1	-1	-1
χ_3	1	-1	-1	1

Each entry in the table is real, so for each δ_g, we use the column corresponding to g, scaled by $\frac{1}{4}$. For example,

$$\delta_{(0,1)} = \frac{1}{4}(\chi_0 + \chi_1 - \chi_2 - \chi_3).$$

For Parseval's identity,

$$\langle \delta_g, \delta_g \rangle = \frac{1}{4} = 4(\frac{1}{16}) = \sum_i |c_i|^2.$$

6. (a) Linear functional: $e_g(cf + h) = (cf + h)(g) = c(f(g)) + h(g) = ce_g(f) + e_g(h)$.

Linear independence: Suppose $E = \sum_g c_g e_g = 0$. Then

$$Ef = \sum_g c_g e_g(f) = \sum c_g f(g) = 0$$

for all f in $L^2(G)$. If we define $f(g) = \overline{c_g}$, then

$$Ef = \sum_g |c_g|^2 = 0$$

and each $c_g = 0$. Thus $E = 0$.

Spanning: If L is a linear functional on $L^2(G)$ then we may represent it in the form $L(f) = \langle f, h\rangle$. Note that

$$Lf = \langle f, h\rangle = \frac{1}{|G|}\sum_g f(g)\overline{h(g)} = \frac{1}{|G|}\sum_g \overline{h(g)}e_g(f).$$

Thus $L = \sum c_g e_g$ with $c_g = \frac{1}{|G|}\overline{h(g)}$.

(b) $\langle f, |G|\delta_g\rangle = \frac{1}{|G|}\sum_{g'} f(g')\overline{|G|\delta_g(g')} = f(g)$.

(c) Orthonormal:

$$\langle \sqrt{|G|}\delta_g, \sqrt{|G|}\delta_{g'}\rangle = \sum_{g''}\delta_g(g'')\overline{\delta_{g'}(g'')} = \begin{cases} 1 & g = g' \\ 0 & \text{otherwise} \end{cases}.$$

Permutation: $(L(g)\delta_{g'})(x) = \delta_{g'}(g^{-1}x)$, so $L(g)\delta_{g'} = \delta_{gg'}$. The scaling by $\sqrt{|G|}$ is preserved by L.

(d) By observation,

$$\sqrt{2}\delta_0 = \frac{\sqrt{2}}{2}(\chi_0 + \chi_1), \qquad \sqrt{2}\delta_1 = \frac{\sqrt{2}}{2}(\chi_0 - \chi_1),$$

and

$$P = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

is unitary.

7. (a) Linear:

$$\begin{aligned} [\pi(\chi)(cf + h)](g) &= \chi(g)(cf + h)(g) \\ &= c\chi(g)f(g) + \chi(g)h(g) \\ &= c[\pi(\chi)f](g) + [\pi(\chi)h](g). \end{aligned}$$

Homomorphism:

$$\pi(\chi\chi')f](g) = \chi(g)\chi'(g)f(g) = \pi(\chi)[\chi' f](g) = [\pi(\chi)[\pi(\chi')f]](g).$$

Unitary:

$$\begin{aligned}\langle \pi(\chi)f, \pi(\chi)h\rangle &= \frac{1}{|G|}\sum_g \chi(g)f(g)\overline{\chi(g)h(g)}\\ &= \frac{1}{|G|}\sum_g f(g)\overline{h(g)} = \langle f, h\rangle.\end{aligned}$$

(b) Note that $\pi(\chi)\delta_g(h) = \chi(h)\delta_g(h)$. This is always zero unless $g = h$, so

$$\pi(\chi)\delta_g = \chi(g)\delta_g = e_g(\chi)\delta_g.$$

Thus the irreducible subrepresentations are one-dimensional with subspace $\mathbb{C}\delta_g$ of type e_g on G^*.

8. In each case, the dimension formula implies that one row is missing, and the first entry is the dimension. For S_3, the third row is (2 a b). Orthonormality gives

$$2^2 + 3|a|^2 + 2|b|^2 = 6, \quad 3a + 2b = -2, \quad -3a + 2b = -2.$$

S_3	e (1)	(12) (3)	(123) (2)
χ_0	1	1	1
χ_1	1	-1	1
π_1	2	0	-1

For D_8, we have last row (2 a b c d). Orthonormality gives $|a|^2+2|b|^2+2|c|^2+2|d|^2 = 4$, and the equations

$$\begin{aligned}a + 2b + 2c + 2d &= -2,\\ a + 2b - 2c - 2d &= -2,\\ a - 2b - 2c + 2d &= -2,\\ a - 2b + 2c - 2d &= -2.\end{aligned}$$

D_8	e (1)	r^2 (1)	r (2)	c (2)	cr (2)
χ_0	1	1	1	1	1
χ_1	1	1	1	-1	-1
χ_2	1	1	-1	-1	1
χ_3	1	1	-1	1	-1
π_1	2	-2	0	0	0

For Q, we have last row (2 a b c d) and a computation similar to D_8. Note that D_8 and Q have the same character tables.

Q	1 (1)	-1 (1)	$\pm i$ (2)	$\pm j$ (2)	$\pm k$ (2)
χ_0	1	1	1	1	1
χ_1	1	1	1	-1	-1
χ_2	1	1	-1	1	-1
χ_3	1	1	-1	-1	1
π_1	2	-2	0	0	0

For A_4, we have last row (3 a b c). Orthonormality gives $4|a|^2 + 4|b|^2 + 3|c|^2 = 3$, and the equations

$$\begin{aligned} 4a + 4b + 3c &= -3, \\ 4a\omega^2 + 4b\omega + 3c &= -3, \\ 4a\omega + 4b\omega^2 + 3c &= -3. \end{aligned}$$

Summing the equations gives $c = -1$ since $1 + \omega + \omega^2 = 0$ as $\omega = e^{2\pi/3}$.

A_4	e (1)	(123) (4)	(132) (4)	(12)(34) (3)
χ_0	1	1	1	1
χ_1	1	ω	ω^2	1
χ_2	1	ω^2	ω	1
π	3	0	0	-1

9. (a) Conjugacy classes: $\{e\}$, $\{r^3\}$, $\{r, r^5\}$, $\{r^2, r^4\}$, $\{c, cr^2, cr^4\}$, $\{cr, cr^3, cr^5\}$.

Center: $Z(D_{12}) = \{e, r^3\}$.

Commutator: $[D_{12}, D_{12}] = \{e, r^2, r^4\}$ and the abelianization $D_{12}/[D_{12}, D_{12}]$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Normal Subgroups: All subgroups of the rotation subgroup $\langle r \rangle$ are normal, as is $\langle c, r^2 \rangle \cong S_3$ and D_{12}.

(b) Since the abelianization is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. Thus there are four characters, and one notes that the abelianization is generated by c and r, each with order 2 in the quotient group.

D_{12}	e (1)	r^3(1)	r (2)	r^2 (2)	c (3)	cr (3)
χ_0	1	1	1	1	1	1
χ_1	1	-1	-1	1	1	-1
χ_2	1	1	1	1	-1	-1
χ_3	1	-1	-1	1	-1	1

(c) The defining representation $(\pi_1, \mathbb{C}^2)$ of D_{12} is generated by

$$\pi_1(c) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \pi_1(r) = \begin{bmatrix} cos(\pi/3) & -sin(\pi/3) \\ sin(\pi/3) & cos(\pi/3) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Checking normal subgroups, we see that $D_{12}/Z(D_{12})$ is isomorphic to S_3, so we may compose the irreducible two-dimensional representation of S_3 with the quotient map. This representation is defined by

$$\pi_2(c) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \pi_2(r) = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

Comparing traces show these are inequivalent.

(d) For reflections, the eigenvalues are 1 and -1, so these have trace 0.

D_{12}	e (1)	r^3(1)	r (2)	r^2 (2)	c (3)	cr (3)
χ_0	1	1	1	1	1	1
χ_1	1	-1	-1	1	1	-1
χ_2	1	1	1	1	-1	-1
χ_3	1	-1	-1	1	-1	1
π_1	2	-2	1	-1	0	0
π_2	2	2	-1	-1	0	0

We leave it to the reader to verify orthonormality of the rows. Also compare the number of conjugacy classes with number of irreducibles.

10. Let $n = 2l+1$. We have relations $r^n = e,\ c^2 = e,\ crc = r^{-1}$.

Conjugacy classes: $\{e\}$, $\{r^{\pm}i\}$ $(0 \leq i \leq l)$, and the reflections $\{cr^k \mid 0 \leq k < n\}$. So there are $l+2$ conjugacy classes. The center is trivial.

Commutator: The subgroup is generated by r^2. Since n is odd, this is the subgroup of all rotations. Thus the abelianization is isomorphic to $\mathbb{Z}/2$, and there are two characters.

Normal Subgroups: All subgroups of the rotation subgroup, and D_{2n} itself.

The remaining l irreducible representations are two-dimensional. In addition to the defining representation $(\pi_1, \mathbb{C}^2)$, we have, for each $1 \leq k \leq l$,

$$\pi_k(c) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \pi_k(r) = \begin{bmatrix} cos(2k\pi/n) & -sin(2k\pi/n) \\ sin(2k\pi/n) & cos(2k\pi/n) \end{bmatrix}.$$

Since

$$\chi_{\pi_k}(r) = Trace(\pi_k(r)) = 2cos(2k\pi/n)$$

is distinct for each k, $\{\pi_k\}$ forms a set of inequivalent representations. Checking the dimension formula,

$$1^2 + 1^2 + l(2^2) = 4l + 2 = 2n = |D_{2n}|.$$

For the character table,

D_{2n}	e (1)	r(2)	$\ldots$	r^l (2)	c (n)
χ_0	1	1	$\ldots$	1	1
χ_1	1	1	$\ldots$	1	-1
π_1	2	$2cos(2\pi/n)$	$\ldots$	$2cos(2l\pi/n)$	0
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
π_l	2	$2cos(2l\pi/n)$	$\ldots$	$2cos(2l^2\pi/n)$	0

To verify orthonormality of rows, we note the equations

$$cos(s)cos(t) = \frac{1}{2}[cos(s+t) + cos(s-t)]$$

and

$$cos(t) = \frac{e^{it} + e^{-it}}{2},$$

and apply Problem 3(c).