INTRO TO GROUP REPS - JULY 30, 2012
SOLUTION SET 8
RT7. FINITE ABELIAN GROUPS 2

1. (a) If we write π as a direct sum of irreducible representations, then $\pi = \bigoplus_{\chi \in G^*} n_\chi \chi$ and

$$\chi_\pi(g) = Trace(\pi(g)) = \sum_{\chi \in G^*} n_\chi \chi.$$

Because $\{\chi\}$ is an orthonormal basis for $L^2(G)$, $\langle \chi_\pi, \chi \rangle = n_\chi$.

(b) With respect to the orthonormal basis $\{\sqrt{|G|}\delta_g\}$ of $L^2(G)$, each $L(g)$ acts as a permutation of the basis vectors without fixed points unless $g = e$. Thus

$$\chi_L(g) = \begin{cases} |G| & g = e \\ 0 & \text{otherwise} \end{cases}.$$

This means $\langle \chi_L, \chi \rangle = 1$ for each χ in G^*.

2. (a) Restricting to $H_1 \cong \mathbb{Z}/2$,

$$\chi_\pi(g) = \begin{cases} 2 & g = e \\ -2 & g = r^2 \end{cases}.$$

Thus $n_{triv} = 0$ and $n_{sgn} = 2$.

Restricting to $H_2 \cong \mathbb{Z}/2$,

$$\chi_\pi(g) = \begin{cases} 2 & g = e \\ 0 & g = c \end{cases}.$$

Thus $n_{triv} = 1$ and $n_{sgn} = 1$.

Restricting to $H_3 \cong \mathbb{Z}/4$,

$$\chi_\pi(g) = \begin{cases} 2 & g = e \\ -2 & g = r^2 \\ 0 & g = r, r^3 \end{cases}.$$

Referring to Solution Set 1, Problem 8(b), $n_{\chi_0} = 0$, $n_{\chi_1} = 1$, $n_{\chi_2} = 0$, and $n_{\chi_3} = 1$.

(b) Restricting to $H_1 \cong \mathbb{Z}/2$,

$$\chi_\pi(g) = \begin{cases} 2 & g = 1 \\ 2 & g = -1 \end{cases}.$$

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Thus $n_{triv} = 2$ and $n_{sgn} = 0$.

Restricting to $H_2 \cong \mathbb{Z}/4$,

$$\chi_\pi(g) = \begin{cases} 2 & g = e \\ -2 & g = -1 \\ 0 & g = \pm i \end{cases}.$$

Referring to Solution Set 1, Problem 8(b), $n_{\chi_0} = 0,\ n_{\chi_1} = 1,\ n_{\chi_2} = 0$, and $n_{\chi_3} = 1$.

(c) Restricting to $H_1 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$,

$$\chi_\pi(g) = \begin{cases} 3 & g = e \\ -1 & g = (12)(34), (13)(24), (14)(23) \end{cases}.$$

Referring to Solution Set 1, Problem 8(b), $n_{\chi_0} = 0,\ n_{\chi_1} = 1, n_{\chi_2} = 1$, and $n_{\chi_3} = 1$.

Restricting to $H_2 \cong \mathbb{Z}/3$,

$$\chi_\pi(g) = \begin{cases} 3 & g = e \\ 0 & g = (123), (132) \end{cases}.$$

Referring to Solution Set 1, Problem 8(b), $n_{\chi_0} = 1,\ n_{\chi_1} = 1$, and $n_{\chi_2} = 1$.

3. (a) Straightforward.

(b)

$$L(g)\delta_{g'}(x) = \delta_{g'}(g^{-1}x) = \begin{cases} 1 & \text{if } x = gg' \\ 0 & \text{otherwise} \end{cases}.$$

$$R({g'}^{-1})\delta_g(x) = \delta_g(x{g'}^{-1}) = \begin{cases} 1 & x = gg' \\ 0 & \text{otherwise} \end{cases}.$$

$$\delta_g * \delta_{g'}(x) = \frac{1}{|G|}\sum_h \delta_g(h)\delta_{g'}(h^{-1}x) = \frac{1}{|G|}\delta_{gg'}(x).$$

In the sum, all terms vanish unless $h = g$ and $h^{-1}x = g'$. Thus the only nonvanishing case is when $x = gg'$.

(c) Since convolution is bilinear, it suffices to verify when each f_i is a character. First note that

$$\chi * \chi' = \chi' * \chi = \begin{cases} \chi & \chi = \chi' \\ 0 & \text{otherwise} \end{cases},$$

showing commutativity. For associativity,

$$(\chi_1 * \chi_2) * \chi_3 = 0 = \chi_1 * (\chi_2 * \chi_3) \quad \text{if any } \chi_i \neq \chi_j.$$

Otherwise

$$(\chi * \chi) * \chi = \chi * \chi = \chi * (\chi * \chi).$$

Alternatively, one can verify these on the basis of delta functions.

(d) $(\delta_0 + 2\delta_1) * (2\delta_0 + \delta_1) = \frac{1}{2}(4\delta_0 + 5\delta_1) = (2\delta_0 + \delta_1) * (\delta_0 + 2\delta_1)$.

$(\delta_0 + 2\delta_1) * ((2\delta_0 + \delta_1) * (2\delta_0)) = (\delta_0 + 2\delta_1) * \frac{1}{2}(4\delta_0 + 2\delta_1) = \frac{1}{4}(8\delta_0 + 10\delta_1)$,
$((\delta_0 + 2\delta_1) * (2\delta_0 + \delta_1)) * (2\delta_0) = \frac{1}{2}(4\delta_0 + 5\delta_1) * (2\delta_0) = \frac{1}{4}(8\delta_0 + 10\delta_1)$.

4. (a)

$$\begin{aligned}(\widehat{cf_1 + f_2})(\chi) &= \langle cf_1 + f_2, \chi\rangle \\ &= c\langle f_1, \chi\rangle + \langle f_2, \chi\rangle \\ &= c\widehat{f_1}(\chi) + \widehat{f_2}(\chi).\end{aligned}$$

(b) First recall that

$$\langle h_1, h_2\rangle_{L^2(B)} = \sum_{\chi \in G^*} h_1(\chi)\overline{h_2(\chi)}.$$

Then

$$\begin{aligned}\langle \widehat{f_1}, \widehat{f_1}\rangle &= \sum_{\chi \in G^*} \widehat{f_1}(\chi)\overline{\widehat{f_1}(\chi)} \\ &= \sum_{\chi \in G^*} \langle f_1, \chi\rangle\overline{\langle f_1, \chi\rangle} \\ &= \langle f_1, f_1\rangle\end{aligned}$$

by Parseval's Identity for $L^2(G)$.

(c)

$$\begin{aligned}(\widehat{f_1 * f_2})(\chi) &= \frac{1}{|G|}\sum_{g \in G}(f_1 * f_2)(g)\overline{\chi(g)} \\ &= \frac{1}{|G|}\sum_{g \in G}\frac{1}{|G|}\sum_{g' \in G} f_1(g')f_2(g'^{-1}g)\overline{\chi(g)} \\ &= \frac{1}{|G|}\sum_{g \in G}\frac{1}{|G|}\sum_{g' \in G} f_1(g')f_2(g)\overline{\chi(g'g)} \\ &= \Big[\frac{1}{|G|}\sum_{g' \in G} f_1(g')\overline{\chi(g')}\Big]\Big[\frac{1}{|G|}\sum_{g \in G} f_2(g)\overline{\chi(g)}\Big] \\ &= \widehat{f_1}(\chi)\widehat{f_2}(\chi).\end{aligned}$$

(d) First

$$\widehat{\chi * \chi'} = \begin{cases}\delta_\chi & \chi = \chi' \\ 0 & \text{otherwise}\end{cases}.$$

On the other hand,

$$\widehat{\chi} \cdot \widehat{\chi'} = \delta_\chi \cdot \delta_{\chi'} = \begin{cases} \delta_\chi & \chi = \chi' \\ 0 & \text{otherwise} \end{cases}.$$

(e)

$$\begin{aligned} \widehat{f_1 * f_2} &= (\delta_0 + 2\widehat{\delta_1) * (2}\delta_0 + \delta_1) \\ &= (2\widehat{\delta_0 + \frac{5}{2}}\delta_1) \\ &= \frac{9}{4}\delta_{triv} - \frac{1}{4}\delta_{sgn}. \end{aligned}$$

$$\begin{aligned} \widehat{f_1} \cdot \widehat{f_2} &= \delta_0 \widehat{+ 2\delta_1} \cdot 2\widehat{\delta_0 + \delta_1} \\ &= (\frac{3}{2}\delta_{triv} - \frac{1}{2}\delta_{sgn}) \cdot (\frac{3}{2}\delta_{triv} + \frac{1}{2}\delta_{sgn}) \\ &= \frac{9}{4}\delta_{triv} - \frac{1}{4}\delta_{sgn}. \end{aligned}$$

5. (a) Noting that $P_{\chi_1} f = \langle f, \chi_1 \rangle \chi_1$ and that $\frac{1}{2}\chi_1(g) = \langle \chi_1, 2\delta_g \rangle$,

$$P_{\chi_1}(2\delta_g) = \frac{1}{2}\chi_1(g^{-1})\chi_1.$$

$$[P_{\chi_1}]_B = \frac{1}{4}\begin{bmatrix} 1 & -i & -1 & i \\ i & 1 & -i & -1 \\ -1 & i & 1 & -i \\ -i & -1 & i & 1 \end{bmatrix}$$

(b) As noted above, $P_{\chi_i}(\delta_1) = \frac{1}{4}\chi_i(3)\chi_i$, so

$$P_{\chi_0}(\delta_1) = \frac{1}{4}\chi_0, \quad P_{\chi_1}(\delta_1) = -\frac{1}{4}i\chi_1, \quad P_{\chi_2}(\delta_1) = -\frac{1}{4}\chi_2, \quad P_{\chi_3}(\delta_1) = \frac{1}{4}i\chi_3.$$

(c) Suppose $f = \sum_\chi c_\chi \chi$. Then

$$\chi(f) = \frac{1}{|G|}\sum_{g \in G} f(g)\chi(g) = c_{\overline{\chi}}.$$

Thus

$$\sum_\chi \chi(f)\overline{\chi(f)} = \sum_{g \in G} |c_{\overline{\chi}}|^2 = \langle f, f \rangle.$$

(d) $(f * f^*)(e) = \frac{1}{|G|}\sum_{g \in G} f(g) f^*(g^{-1}e) = \frac{1}{|G|}\sum_{g \in G} f(g)\overline{f(g)} = \langle f, f \rangle$.

6. (a)

$$\begin{aligned}
\pi(L(g)f)v &= \frac{1}{|G|}\sum_{g'\in G} L(g)f(g')\pi(g')v \\
&= \frac{1}{|G|}\sum_{g'\in G} f(g^{-1}g')\pi(g')v \\
&= \frac{1}{|G|}\sum_{g'\in G} f(g')\pi(gg')v = \pi(g)\pi(f)v.
\end{aligned}$$

$$\begin{aligned}
\pi(R(g)f)v &= \frac{1}{|G|}\sum_{g'\in G} R(g)f(g')\pi(g')v \\
&= \frac{1}{|G|}\sum_{g'\in G} f(g'g)\pi(g')v \\
&= \frac{1}{|G|}\sum_{g'\in G} f(g')\pi(g'g^{-1})v = \pi(f)\pi(g^{-1})v.
\end{aligned}$$

(b)

$$\begin{aligned}
\pi(f*h)v &= \frac{1}{|G|}\sum_{g\in G}(f*h)(g)\pi(g)v \\
&= \frac{1}{|G|}\sum_{g\in G}\frac{1}{|G|}\sum_{g'\in G} f(g')h(g'^{-1}g)\pi(g)v \\
&= \frac{1}{|G|}\sum_{g\in G}\frac{1}{|G|}\sum_{g'\in G} f(g')h(g)\pi(g')\pi(g)v \\
&= \big[\frac{1}{|G|}\sum_{g'\in G} f(g')\pi(g')\big]\big[\frac{1}{|G|}\sum_{g\in G} h(g)\pi(g)\big]v \\
&= \pi(f)\pi(h)v
\end{aligned}$$

By the rule for convolution of characters, we have $f*h = \sum_\chi a_\chi b_\chi \chi$. Thus $[\pi(f*h)]_B$ is diagonal with diagonal entries $a_\chi b_\chi$. Since $[\pi(f)]_B$ (resp. $[\pi(h)]_B$) is diagonal with entries a_χ (resp. b_χ), the result follows.

(c)

$$\begin{aligned}\langle \pi(f^*)v, w\rangle &= \langle \frac{1}{|G|}\sum_{g\in G} f^*(g)\pi(g)v, w\rangle \\ &= \langle \frac{1}{|G|}\sum_{g\in G} \overline{f(g^{-1})}\pi(g)v, w\rangle \\ &= \langle v, \frac{1}{|G|}\sum_{g\in G} f(g^{-1})\pi(g^{-1})w\rangle \\ &= \langle v, \pi(f)w\rangle\end{aligned}$$

7. (a)

$$\begin{aligned}P_{\chi_0} &= \pi(\overline{\chi_0}) = \frac{1}{4}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}0&-1\\1&0\end{bmatrix}+\begin{bmatrix}-1&0\\0&-1\end{bmatrix}+\begin{bmatrix}0&1\\-1&0\end{bmatrix}\right)=0,\\ P_{\chi_1} &= \pi(\overline{\chi_1}) = \frac{1}{4}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}-i\begin{bmatrix}0&-1\\1&0\end{bmatrix}-\begin{bmatrix}-1&0\\0&-1\end{bmatrix}+i\begin{bmatrix}0&1\\-1&0\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}1&i\\-i&1\end{bmatrix},\\ P_{\chi_2} &= \pi(\overline{\chi_2}) = \frac{1}{4}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}0&-1\\1&0\end{bmatrix}+\begin{bmatrix}-1&0\\0&-1\end{bmatrix}-\begin{bmatrix}0&1\\-1&0\end{bmatrix}\right)=0,\\ P_{\chi_3} &= \pi(\overline{\chi_3}) = \frac{1}{4}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}+i\begin{bmatrix}0&-1\\1&0\end{bmatrix}-\begin{bmatrix}-1&0\\0&-1\end{bmatrix}-i\begin{bmatrix}0&1\\-1&0\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}1&-i\\i&1\end{bmatrix}.\end{aligned}$$

(b) That $P^2 = P$, $P^* = P$, and $P_{\chi_1} + P_{\chi_3} = I$ are straightforward.

(c) The χ_1 type (resp. χ_3) is spanned by $(1, i)$ (resp. $(1, -i)$). These form a joint eigenbasis for $\{\pi(r^i)\}$.

8. First

$$\langle v + w, v + w\rangle = \langle v, v\rangle + \langle w, w\rangle + \langle v, w\rangle + \langle w, v\rangle.$$

By the Hermitian property, $\langle v, w\rangle = \overline{\langle w, v\rangle}$, so

$$Re(\langle v, w\rangle) = \frac{1}{2}(||v + w||^2 - ||v||^2 - ||w||^2).$$

If $z = x + iy$, then $Re(iz) = -y = -Im(z)$, so

$$\begin{aligned}Im(\langle v, w\rangle) &= -Re(i\langle v, w\rangle) = -Re(\langle iv, w\rangle \\ &= -\frac{1}{2}(||iv + w||^2 - ||v||^2 - ||w||^2 \\ &= \frac{1}{2}(||v||^2 + ||w||^2 - ||iv + w||^2)\end{aligned}$$

9. (a) Conjugacy classes with element orders:

$$\{e\}(1),\ \ \{y^2\}(2),\ \ \{x,\ x^2\}(3),\ \ \{y^2x,\ y^2x^2\}(6),\ \{y,\ yx,\ yx^2\}(4),\ \ \{y^3,\ y^3x,\ y^3x^2\}(4).$$

Thus there are six conjugacy classes.

Center: $Z(G) = \{e, y^2\}$.

Commutator: $[G, G] = \{e, x, x^2\}$.

Normal Subgroups: $\{e\}$, $\{e, y^2\}$, $\langle x\rangle$, $\langle y^2x\rangle$, G.

(b) The abelianization is isomorphic to $\mathbb{Z}/4$, generated by y. Thus there are four characters, and the remaining irreducible classes have dimension 2. Thus

$$1^2 + 1^2 + 1^2 + 1^2 + 2^2 + 2^2 = 12.$$

If we quotient by the center, the quotient is isomorphic to S_3, and we may extend the irreducible two-dimensional representation π_1 of S_3 to G. The remaining row is completed using orthonormality or by considering $\chi_1 \otimes \pi_1$. Note that π_2 corresponds to a faithful representation. For example, the matrices

$$\chi_1(y)\pi_1(y) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \chi_1(x)\pi_1(x) = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

satisfy the defining relations for G.

G	e (1)	y^2 (1)	x (2)	y^2x (2)	y (3)	y^3 (3)
χ_0	1	1	1	1	1	1
χ_1	1	-1	1	-1	i	-i
χ_2	1	1	1	1	-1	-1
χ_3	1	-1	1	-1	-i	i
π_1	2	2	-1	-1	0	0
π_2	2	-2	-1	1	0	0

10. Conjugacy classes: $\{e\}$, $\{r^l\}$, $\{r^{\pm i}\}$ $(1 \leq i < l)$, $\{cr^{even}\}$, $\{cr^{odd}\}$. Thus there are $l + 3$ conjugacy classes.

Center: $Z(D_{2n}) = \{e, r^l\}$.

Commutator: $[D_{2n}, D_{2n}] = \langle r^2\rangle$. So the abelianization is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, generated by c and r. Thus there are 4 characters.

Normal Subgroups: All subgroups of the rotation subgroup $\langle r\rangle$, $\langle c, r^2\rangle$, and D_{2n} itself.

The remaining $l - 1$ irreducible representations are two-dimensional. In addition to the defining representation $(\pi_1, \mathbb{C}^2)$, we have, for each $1 \leq k < l$,

$$\pi_k(c) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \pi_k(r) = \begin{bmatrix} cos(2k\pi/n) & -sin(2k\pi/n) \\ sin(2k\pi/n) & cos(2k\pi/n) \end{bmatrix}.$$

Since

$$\chi_{\pi_k}(r) = Trace(\pi_k(r)) = 2cos(2k\pi/n)$$

is distinct for each k, $\{\pi_k\}$ forms a set of inequivalent representations. Checking the dimension formula,

$$1^2 + 1^2 + 1^2 + 1^2 + (l-1)(2^2) = 4l = 2n = |D_{2n}|.$$

For the character table,

D_{2n}	e (1)	r (2)	...	r^{l-1} (2)	r^l (1)	c ($n/2$)	cr ($n/2$)
χ_0	1	1	...	1	1	1	1
χ_1	1	-1	...	$(-1)^{l-1}$	$(-1)^l$	1	-1
χ_2	1	1	...	1	1	-1	-1
χ_3	1	-1	...	$(-1)^{l-1}$	$(-1)^l$	-1	1
π_1	2	$2cos(2\pi/n)$	...	$2cos(2(l-1)\pi/n)$	-2	0	0
...	...	...	...	...	...	...	...
π_{l-1}	2	$2cos(2(l-1)\pi/n)$	...	$2cos(2(l-1)^2\pi/n)$	-2	0	0

Orthogonality of rows follows in a manner similar to Solution Set 7, Problem 10.