INTRO TO GROUP REPS - AUGUST 6, 2012
SOLUTION SET 9
RT8.1. SCHUR ORTHOGONALITY RELATIONS

1. (a) If M is the matrix associated to the character table, then $MDM^* = |G|I$, where D is diagonal with entries equal to the number of elements in the corresponding conjugacy class. Moving M^* to the right-hand side and back, we have $M^*M = |G|D^{-1}$. Since the matrix on the right-hand side is diagonal, the columns of M are orthogonal, and the length squared of the i-th column equals $|G|/|C_x|$, where C_x is the conjugacy class representing the i-th column.

(b) Straighforward.

2. (a) By Problem 1(a),

$$det(MDM^*) = det(|G|I) = |G|^N,$$

where N equals the number of conjugacy classes in G. Thus

$$|det(M)|^2 = |G|^N/C,$$

where $C = det(D)$ is the product of the orders of conjugacy classes in G. Note that the right-hand side is also equal to the product of orders of centralizers $C(x)$, where we choose one representative x for each conjugacy class.

(b) For S_3, $|det(M)|^2 = 36 = 6^3/6 = |G|^3/6$.

For A_4, $|det(M)|^2 = 3 \cdot 12^2 = 12^4/48$.

If G is abelian, then $|det(M)|^2 = |G|^{|G|}$.

3. (a) For g' in G,

$$\begin{aligned} \pi'(g')L &= \frac{1}{|G|}\sum_{g\in G}\pi'(g')\pi'(g^{-1})L\pi(g) \\ &= \frac{1}{|G|}\sum_{g\in G}\pi'(g'g^{-1})L\pi(g) \\ &= \frac{1}{|G|}\sum_{g\in G}\pi'(g^{-1})L\pi(gg') = L\pi(g'). \end{aligned}$$

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(b) By Schur's Lemma, $L' = 0$. Now

$$0 = \langle L'u, u'\rangle = \frac{1}{|G|}\sum_{g\in G}\langle \pi'(g^{-1})\langle \pi(g)u, v\rangle v', u'\rangle = \frac{1}{|G|}\sum_{g\in G}\langle \pi(g)u, v\rangle\overline{\langle \pi'(g)u', v'\rangle}.$$

(c) By Schur's Lemma, $L' = cI$ for some constant c. Let $\{u_i\}$ be an orthonormal basis for V. Since the trace of L is unaffected by conjugation by $\pi(g)$,

$$\begin{aligned} Trace(L') &= Trace(L) = \sum_i \langle Lu_i, u_i\rangle \\ &= \sum_i \langle u_i, v\rangle\langle v', u_i\rangle \\ &= \langle \sum_i \langle v', u_i\rangle u_i, v\rangle \\ &= \langle v', v\rangle = \overline{\langle v, v'\rangle}. \end{aligned}$$

Thus $Trace(L') = c\ dim(V) = \overline{\langle v, v'\rangle}$. Unraveling as in part (b),

$$\frac{1}{|G|}\sum_g \langle \pi(g)u, v\rangle\overline{\langle \pi(g)u', v'\rangle} = c\langle u, u'\rangle = \frac{1}{dim(V)}\langle u, u'\rangle\overline{\langle v, v'\rangle}.$$

4. (a) We use the standard basis for $\mathbb{C}^2$. Recall that $\phi_{ij}(g) = \langle \pi(g)e_i, e_j\rangle$ is the (j,i)-th entry of $\pi(g)$.

D_8	e	r	r^2	r^3	c	cr	cr^2	cr^3
χ_0	1	1	1	1	1	1	1	1
χ_1	1	1	1	1	-1	-1	-1	-1
χ_2	1	-1	1	-1	-1	1	-1	1
χ_3	1	-1	1	-1	1	-1	1	-1
$\sqrt{2}\phi_{11}$	$\sqrt{2}$	0	$-\sqrt{2}$	0	$\sqrt{2}$	0	$-\sqrt{2}$	0
$\sqrt{2}\phi_{21}$	0	$-\sqrt{2}$	0	$\sqrt{2}$	0	$-\sqrt{2}$	0	$\sqrt{2}$
$\sqrt{2}\phi_{12}$	0	$\sqrt{2}$	0	$-\sqrt{2}$	0	$-\sqrt{2}$	0	$\sqrt{2}$
$\sqrt{2}\phi_{22}$	$\sqrt{2}$	0	$-\sqrt{2}$	0	$-\sqrt{2}$	0	$\sqrt{2}$	0

Again we use the standard basis of $\mathbb{C}^2$.

Q	1	-1	i	$-i$	j	$-j$	k	$-k$
χ_0	1	1	1	1	1	1	1	1
χ_1	1	1	1	1	-1	-1	-1	-1
χ_2	1	1	-1	-1	1	1	-1	-1
χ_3	1	1	-1	-1	-1	-1	1	1
$\sqrt{2}\phi_{11}$	$\sqrt{2}$	$-\sqrt{2}$	0	0	$\sqrt{2}i$	$-\sqrt{2}i$	0	0
$\sqrt{2}\phi_{21}$	0	0	$\sqrt{2}$	$-\sqrt{2}$	0	0	$-\sqrt{2}i$	$\sqrt{2}i$
$\sqrt{2}\phi_{12}$	0	0	$-\sqrt{2}$	$\sqrt{2}$	0	0	$-\sqrt{2}i$	$\sqrt{2}i$
$\sqrt{2}\phi_{22}$	$\sqrt{2}$	$-\sqrt{2}$	0	0	$-\sqrt{2}i$	$\sqrt{2}i$	0	0

(b) Fourier's Trick:

$$\begin{aligned} f &= \delta_r + \delta_c \\ &= \frac{1}{8}(\chi_0 + \chi_1 - \chi_2 - \chi_3 - 2\phi_{21} + 2\phi_{12}) + \frac{1}{8}(\chi_0 - \chi_1 - \chi_2 + \chi_3 + 2\phi_{11} - 2\phi_{22}) \\ &= \frac{1}{4}\chi_0 - \frac{1}{4}\chi_2 + \frac{\sqrt{2}}{8}(\sqrt{2}\phi_{11} - \sqrt{2}\phi_{21} + \sqrt{2}\phi_{12} - \sqrt{2}\phi_{22}) \end{aligned}$$

Parseval's Identity: $\frac{1}{4} = \frac{1}{16} + \frac{1}{16} + 4(\frac{1}{32})$.

(c) Fourier's Trick:

$$\begin{aligned} f &= 2\delta_i - \delta_j \\ &= \frac{1}{4}(\chi_0 + \chi_1 - \chi_2 - \chi_3 + 2\phi_{21} - 2\phi_{12}) - \frac{1}{8}(\chi_0 - \chi_1 + \chi_2 - \chi_3 + 2i\phi_{11} - 2i\phi_{22}) \\ &= \frac{1}{8}\chi_0 + \frac{3}{8}\chi_1 - \frac{3}{8}\chi_2 - \frac{1}{8}\chi_3 + \frac{\sqrt{2}}{4}(\sqrt{2}\phi_{21} - \sqrt{2}\phi_{12}) - \frac{\sqrt{2}i}{8}(\sqrt{2}\phi_{11} - \sqrt{2}\phi_{22}) \end{aligned}$$

Parseval's Identity: $\frac{5}{8} = \frac{2}{64} + 2(\frac{9}{64}) + 2(\frac{2}{16}) + 2(\frac{2}{64})$.

5. (a) Since $Trace(\pi \oplus \pi') = Trace(\pi) + Trace(\pi')$, we have

$$\chi_\pi(g) = \sum_{\sigma'} n_{\sigma'}\chi_{\sigma'}(g),$$

where σ' ranges over a set of representatives for each irreducible representation.

By orthonormalty of characters,

$$\langle \chi_\pi, \chi_\sigma \rangle = \langle \sum_{\sigma'} n_{\sigma'}\chi_{\sigma'}, \chi_\sigma \rangle = \sum_{\sigma'} n_{\sigma'} \langle \chi_{\sigma'}, \chi_\sigma \rangle = n_\sigma.$$

(b) Again by orthonormalty of characters,

$$\langle \chi_\pi, \chi_\pi \rangle = \langle \sum_{\sigma'} n_{\sigma'}\chi_{\sigma'}, \sum_{\sigma'} n_{\sigma'}\chi_{\sigma'} \rangle = \sum_{\sigma} n_\sigma^2.$$

(c) Orthonormality of characters show that irreducible characters have length squared 1. The other direction follows from part (b).

6. (a) This is essentially the triangle equality for complex numbers. We prove by induction on the number of terms. For the base case, $|\omega_1| = 1$.

Suppose the statement holds when there are n terms. Suppose $S_n = \sum_{k=1}^{n} \omega_k = re^{i\theta}$. Multiplying S_n and ω_{n+1} by $e^{-i\theta}$, we may assume S_n is real and positive. Now

$$S_{n+1} = S_n + \omega_{n+1} = r + cos(\theta_{n+1}) + isin(\theta_{n+1}),$$

which has length squared

$$(r + cos(\theta_{n+1}))^2 + sin^2(\theta_{n+1}) \leq (r+1)^2.$$

Thus $|S_{n+1}| \leq r + 1 \leq n + 1$ by the induction hypothesis.

If $|S_{n+1}| = n + 1$, then $S_n = r = n$. By the induction hypothesis, each $\omega_k = 1$. Thus $\omega_{k+1} = 1$ also.

(b) The eigenvalues of $\pi(g)$ are roots of unity, and, by part (a), are all equal. Thus $\pi(g)$ is a scalar multiple of the identity since $\pi(g)$ is diagonalizable.

Now choose any faithful representation π' of G. By full reducibility, there exists a basis B that π' is represented by block diagonal matrices (same size for all g' in G). Now $\pi'(g)$ is a scalar multiple of the identity for each block, so $\pi'(g)$ commutes with each $\pi'(g')$. Because π' is faithful, $\pi'(g')$ is in the center of $\pi'(G) \cong G$.

(c) In the abelian case, all elements of G are in the center. For irreducibes, $|dim(V)| = 1 = |\pi(g)|$ always.

7. (a) By Problem 8(a), each eigenvalue of $\pi(g)$ is 1, so $\pi(g) = I$. Thus g is in the kernel of π (possibly only $\{e\}$). Since π is not trivial, the kernel is not all of G.

(b) If π is faithful, $Ker(\pi) = \{e\}$. If $\chi_\pi(g) = dim(V)$, then the eigenvalues of $\pi(g)$ equal 1 and g is in the kernel of π. In the other direction, part (a) implies that the kernel of π is $\{e\}$.

(c) Since G has only $\{e\}$ and G as normal subgroups, every nontrivial representation is faithful, and one applies part (b).

8. (a) To find the character of π, we note the number of fixed points for an element in each class. The length squared of this character is $\frac{1}{24}(16 + 6(4) + 8) = 2$, so there are two inequivalent irreducible subrepresentations. One of these is immediately seen to be trivial, so we may subtract ones to get the character of π_1.

G	e (1)	(12) (6)	(123) (8)	(1234) (6)	(12)(34) (3)
χ_{triv}	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
χ_{π}	4	2	1	0	0
χ_{π_1}	3	1	0	-1	-1

(b) G/N has six elements; if abelian, then the commutator would be contained in N and smaller than A_4. So G/N is isomorphic to S_3, and we extend the irreducible two-dimensional representation to S_4.

S_4	e (1)	(12) (6)	(123) (8)	(1234) (6)	(12)(34) (3)
χ_{triv}	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
π_1	3	1	0	-1	-1
π_2	2	0	-1	0	2

(c) Since there are five irreducible classes, we finish with

S_4	e (1)	(12) (6)	(123) (8)	(1234) (6)	(12)(34) (3)
χ_{triv}	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
π_1	3	1	0	-1	-1
π_2	2	0	-1	0	2
$\chi_{sgn} \otimes \pi_1$	3	-1	0	1	-1

(d) π_1 and $\chi_{sgn} \otimes \pi$ are faithful. If any π has a kernel, some non-trivial conjugacy class must have $\chi_\pi(g) = dim(V)$.

If we restrict to A_4, we have

A_4	e (1)	(123) (4)	(132) (4)	(12)(34) (3)
χ_{triv}	1	1	1	1
χ_{sgn}	1	1	1	1
π_1	3	0	0	-1
π_2	2	-1	-1	2
$\chi_{sgn} \otimes \pi_1$	3	0	0	-1

π_2 has two irreducible subrepresentations for A_4. Since the trivial type does not occur, each non-trivial character occurs exactly once.

(e) Recall that the character of $\pi \otimes \pi'$ is $\chi_\pi \cdot \chi_{\pi'}$. We compute χ_π^2 :

S_4	e (1)	(12) (6)	(123) (8)	(1234) (6)	(12)(34) (3)
$\pi_1 \otimes \pi_1$	9	1	0	1	1

The length squared of the character is $\frac{1}{24}(81+6+6+3) = 4$, and the trivial type occurs with multiplicity one ($\frac{1}{24}(9+6+6+3) = 1$). Thus there are four irreducible types in the decomposition by considering sums of squares equal to 4. These are $\chi_{triv}, \pi_1, \pi_2$, and $\chi_{sgn} \otimes \pi_1$. Checking dimensions, $3^2 = 1+3+2+3$.

9. We have the following table, noting that we multiply characters when tensoring.

S_3	e (1)	(12) (3)	(123) (2)
χ_{triv}	1	1	1
χ_{sgn}	1	-1	1
π	2	0	-1
$\otimes^N \pi$	2^N	0	$(-1)^N$

For the trivial and sgn types, the multiplicities are equal and

$$n_{triv} = n_{sgn} = \frac{1}{6}(2^N + 2(-1)^N) = \frac{1}{3}(2^{N-1} + (-1)^N).$$

For type π,

$$n_\pi = \frac{1}{6}(2^{N+1} + 2(-1)^{N+1}) = \frac{1}{3}(2^N + (-1)^{N+1}).$$

Note that these multiplicities are always integers. Checking dimensions,

$$n_{triv} + n_{sgn} + 2n_\pi = \frac{2}{3}(2^{N-1} + (-1)^N) + \frac{2}{3}(2^N + (-1)^{N+1}) = \frac{2^N}{3} + \frac{2^{N+1}}{3} = 2^N.$$

10. (a) First the sum of squares of dimensions equals 21, so the missing dimensions must be 3 and 3. Then we note that the lengths squared of the last two columns are $|G|/|C_x|$, so the missing entries in these columns are zero.

For the remaining entries, the lengths squared of the columns forces each entry to have $|\chi(g)|^2 = 2$. By row orthogonality,

$$b + c = -1, \; n + p = -1.$$

There are no real solutions with these conditions, so we must have $c = \overline{b}$ and $p = \overline{n}$. Since $b + \overline{b} = -1$, $Re(b) = -\frac{1}{2}$. Now, with $b = x + iy$,

$$2 = |b|^2 = \frac{1}{4} + y^2 \quad \text{implies} \quad y = \pm\frac{\sqrt{7}}{2}.$$

Thus we have the character of π_1, and π_2 is the complex conjugate representation of π_1. For the table, let $\tau = -\frac{1}{2} + \frac{\sqrt{7}}{2}i$.

G	e (1)	x (3)	x^{-1} (3)	y (7)	y^{-1} (7)
χ_0	1	1	1	1	1
χ_1	1	1	1	ω	ω^2
χ_2	1	1	1	ω^2	ω
π_1	3	τ	$\overline{\tau}$	0	0
π_2	3	$\overline{\tau}$	τ	0	0

(b) To find the character of π, we must count the fixed points an element in each class. Since x has order 7, either all or no points are fixed in a seven element set. In this case, there are no fixed points.

We compute the orbits under y: $\{0\}$, $\{1, 2, 4\}$, $\{3, 6, 5\}$. Thus there is exactly one fixed point, and this also holds for y^{-1}. Immediately we may subtract a trivial type from the character of π. Using characters, one either computes or observes that $\pi' = \pi_1 \oplus \pi_2$.

G	e (1)	x (3)	x^{-1} (3)	y (7)	y^{-1} (7)
π	7	0	0	1	1
π'	6	-1	-1	0	0