INTRO TO GROUP REPS - AUGUST 13, 2012
SOLUTION SET 10
RT8. FINITE GROUPS 2

1. Proof 1: Let W be an invariant subspace under π'. Then $\pi'(g)w$ is in W for all w in W and g in G. By definition, $\pi'(g)w = \pi(gN)w$, so W is invariant under π. Since π is irreducible, W is either 0 or V, and π' is irreducible.

Proof 2: Suppose A in $Hom_{\mathbb{C}}(V, V)$ and commutes with each $\pi'(g)$. We show that A is a scalar multiple of the identity on V. Since $\pi'(N) = 1$,

$$\pi(gN)A = \pi'(g)A = A\pi'(g) = A\pi(gN).$$

Since π is irreducible, A is a scalar by Schur's Lemma, and the result follows from the converse of Schur's Lemma for π'.

Proof 3: We show that $\langle \chi_{\pi'}, \chi_{\pi'} \rangle = 1$. By irreducibility, $\sum_{gN \in G/N} |\chi_\pi(gN)|^2 = |G/N|$. First note that, since $\pi'(gn) = \pi'(g)$,

$$\chi_{\pi'}(g) = \chi_\pi(gN).$$

If we choose a set of representatives $\{g'\}$ for each coset in G/N, then each element of G may be uniquely represented as some $g'n$ with n in N. Thus

$$\sum_{g \in G} |\chi_{\pi'}(g)|^2 = \sum_{n \in N} \sum_{g'} |\chi_\pi(g'n)|^2 = |N| \sum_{g'} |\chi_\pi(g'N)|^2 = |N|(|G/N|) = |G|.$$

2. We use Fourier's Trick with the character basis:

$$\begin{aligned}
\langle e_e, \chi_{triv} \rangle &= \frac{1}{6}(1 + 0 + 0) = \frac{1}{6}, \\
\langle e_e, \chi_{sgn} \rangle &= \frac{1}{6}(1 + 0 + 0) = \frac{1}{6}, \\
\langle e_e, \chi_{\pi_2} \rangle &= \frac{1}{6}(2 + 0 + 0) = \frac{1}{3}, \\
FT : e_e &= \frac{1}{6}\chi_{triv} + \frac{1}{6}\chi_{sgn} + \frac{1}{3}\chi_{\pi_2}, \\
PI : \frac{1}{6} &= \frac{1}{36} + \frac{1}{36} + \frac{1}{9}.
\end{aligned}$$

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$$\begin{aligned}
\langle e_{(12)}, \chi_{triv} \rangle &= \frac{1}{6}(0+3+0) = \frac{1}{2}, \\
\langle e_{(12)}, \chi_{sgn} \rangle &= \frac{1}{6}(0-3+0) = -\frac{1}{2}, \\
\langle e_{(12)}, \chi_{\pi_2} \rangle &= \frac{1}{6}(0+0+0) = 0, \\
FT : e_{(12)} &= \frac{1}{2}\chi_{triv} - \frac{1}{2}\chi_{sgn} + 0\chi_{\pi_2}, \\
PI : \frac{1}{2} &= \frac{1}{4} + \frac{1}{4} + 0.
\end{aligned}$$

$$\begin{aligned}
\langle e_{(123)}, \chi_{triv} \rangle &= \frac{1}{6}(0+0+2) = \frac{1}{3}, \\
\langle e_{(123)}, \chi_{sgn} \rangle &= \frac{1}{6}(0+0+2) = \frac{1}{3}, \\
\langle e_{(123)}, \chi_{\pi_2} \rangle &= \frac{1}{6}(0+0-2) = -\frac{1}{3}, \\
FT : e_{(123)} &= \frac{1}{3}\chi_{triv} + \frac{1}{3}\chi_{sgn} - \frac{1}{3}\chi_{\pi_2}, \\
PI : \frac{1}{3} &= \frac{1}{9} + \frac{1}{9} + \frac{1}{9}.
\end{aligned}$$

3. First recall that $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$. For S_n, the inverse of each permutation has the same cycle structure and thus belongs to the same conjugacy class. Hence $\chi_\pi(g) = \chi_\pi(g^{-1})$. For D_{2n}, a similar result holds by checking classes. For instance, when n is odd, the reflections, all self-inverse, form a single class, and the nontrivial rotations pair into conjugacy classes $\{r^{\pm k}\}$.

To obtain non-real entries, we must have an x such that x^{-1} belongs to a different conjugacy class. Aside from abelian groups with elements with order greater than 2, see also the semidirect product of $\mathbb{Z}/3$ on $\mathbb{Z}/7$ and $SL(3, \mathbb{Z}/2)$. This condition is not sufficient as seen with A_4 or A_5. (Next assignment.)

4. (a) Let $\{u_i\}$ be an orthonormal basis for V.

$$
\begin{aligned}
\langle \phi_{u,v}, \chi_\pi \rangle &= \sum_i \langle \phi_{u,v}, \phi_{u_i,u_i} \rangle \\
&= \frac{1}{d_\pi} \sum_i \langle u, u_i \rangle \overline{\langle v, u_i \rangle} \\
&= \frac{1}{d_\pi} \langle u, \sum_i \langle v, u_i \rangle u_i \rangle \\
&= \frac{1}{d_\pi} \langle u, v \rangle
\end{aligned}
$$

Recall that $d_\pi \phi_{u,v} * \phi_{u',v'} = \langle u, v' \rangle \phi_{u',v}$. Then

$$
\begin{aligned}
d_\pi \phi_{u,v} * \chi_\pi(g) &= \sum_i d_\pi \phi_{u,v} * \phi_{u_i,u_i}(g) \\
&= \sum_i \langle u, u_i \rangle \phi_{u_i,v}(g) \\
&= \sum_i \langle u, u_i \rangle \langle \pi(g) u_i, v \rangle \\
&= \langle u, \pi(g)^{-1} v \rangle = \phi_{u,v}(g).
\end{aligned}
$$

(b) It is enough to check on matrix coefficients, and we may further assume all belong to the irreducible representation (π, V). Then

$$d_\pi^2 (\phi_{u_1,v_1} * \phi_{u_2,v_2}) * \phi_{u_3,v_3} = d_\pi \langle u_1, v_2 \rangle \phi_{u_2,v_1} * \phi_{u_3,v_3} = \langle u_1, v_2 \rangle \langle u_2, v_3 \rangle \phi_{u_3,v_1}.$$

On the other hand,

$$d_\pi^2 \phi_{u_1,v_1} * (\phi_{u_2,v_2} * \phi_{u_3,v_3}) = d_\pi \phi_{u_1,v_1} * \langle u_2, v_3 \rangle \phi_{u_3,v_2} = \langle u_2, v_3 \rangle \langle u_1, v_2 \rangle \phi_{u_3,v_1}.$$

(c) For the first part, it is enough to check when f is a matrix coefficient and h is a character. By orthogonality, we may assume both belong to the irreducible representation π. This follows from part (a). For the second part, we may assume f and h are irreducible characters. By orthogonality, we may assume $f = h = \chi_\pi$ for irreducible π. The result follows from $d_\pi \chi_\pi * \chi_\pi = \chi_\pi$.

5. (a) There are $|G|$ terms in the sum $\sum_g |\chi_\pi(g)|^2 = |G|$. Since $\chi_\pi(e) = dim(V) > 1$, there must be at least $d_\pi^2 - 1$ zeros in the sum.

Characters of S_n are always integer-valued, but the proof requires Galois theory.

(b) Straightforward check. The usual proof requires Galois theory.

6. (a) We use the formula for multiplicities $n_{\sigma'} = \langle \chi_{\pi \otimes \pi}, \chi_{\sigma'} \rangle$.

S_3	e (1)	(12) (3)	(123) (2)
χ_{triv}	1	1	1
χ_{sgn}	1	-1	1
π	2	0	-1
$\pi \otimes \pi$	4	0	1

Now $n_{triv} = 1$, $n_{sgn} = 1$, $n_\pi = 1$.

(b) With respect to the standard basis, we define π on generators as follows:

$$\pi(123) = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \qquad \pi(23) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We use the orthonormal basis $B \otimes B = \{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$ for $\mathbb{C}^2 \otimes \mathbb{C}^2$.

With respect to $B \otimes B$, we have matrices for generators:

$$(\pi \otimes \pi)(123) = \frac{1}{4} \begin{bmatrix} 1 & \sqrt{3} & \sqrt{3} & 3 \\ -\sqrt{3} & 1 & -3 & \sqrt{3} \\ -\sqrt{3} & -3 & 1 & \sqrt{3} \\ 3 & \sqrt{3} & -\sqrt{3} & 1 \end{bmatrix}, \qquad (\pi \otimes \pi)((23) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In addition, $\pi(132) = \pi(123)^T$, and $\pi(13)$ and $\pi(12)$ are computed from $(13) = (23)(123)$ and $(12) = (23)(132)$.

We compute each projection operator

$$P_{\pi'} = d_\sigma \sigma(\overline{\chi_{\pi'}}) = \frac{d_\sigma}{6} \sum_g \overline{\chi_{\pi'}(g)} (\pi \otimes \pi)(g).$$

$$P_{triv} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}, \qquad P_{sgn} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad , P_{\pi_2} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Note that each $P^2 = 0$, $P = P^*, \sum P = I$, etc.

(c) Reading off the columns in part (b) and rescaling,

(1) Trivial: $e_1 \otimes e_1 + e_2 \otimes e_2$,
(2) Sgn: $e_1 \otimes e_2 - e_2 \otimes e_1$,
(3) $\pi_2 :$ $e_1 \otimes e_1 - e_2 \otimes e_2$, $e_1 \otimes e_2 + e_2 \otimes e_1$.

Note that these basis vectors are mutually orthogonal using the Hermitian inner product on tensors.

7. (a) See Problem 8 for general proof. Here the trivial type occurs exactly once, and there are two irreducible types, each occurring with multiplicity 1.

(b) $n = 3$: Classes are represented by $e, (12), (123)$ with fixed point counts 3, 1, 0. Thus $|G| = 6 = 3 + 3(1) + 0(2)$.

$n = 4$: Classes are represented by $e, (12), (123), (1234), (12)(34)$ with fixed point counts 4, 2, 1, 0, 0. Thus

$$|G| = 24 = 4 + 6(2) + 1(4) + 6(0) + 3(0).$$

$n = 5$: Classes are represented by $e, (12), (123), (1234), (12345), (12)(34), (123)(45)$ with fixed point counts 5, 3, 2, 1, 0, 1, 0. Thus

$$|G| = 120 = 5 + 10(3) + 20(2) + 30(1) + 24(0) + 15(1) + 20(0).$$

8. (a) For each g in G, $\chi_\pi(g) = f(g)$, the number of points fixed by g. We have also seen that the multiplicity of the trivial representation in π is equal to the number of orbits in X by G. A basis for the trivial types are given by $\{e(x)\}$ where

$$e(x) = \sum_{g\in G} e_{gx}.$$

Now

$$n_X = \langle \chi_\pi, \chi_{triv} \rangle = \frac{1}{|G|} \sum_{g\in G} \chi_\pi(g) = \frac{1}{|G|} \sum_{g\in G} f(g).$$

On the other hand,

$$\sum_\sigma n_\sigma^2 = \langle \chi_\pi, \chi_\pi \rangle = \frac{1}{|G|} \sum_{g\in G} |\chi_\pi(g)|^2 = \frac{1}{|G|} \sum_{g\in G} f(g)^2.$$

(b) When n is odd, the identity has n fixed points, each rotation is fixed point free, and each reflection has one fixed point. Thus

$$|G| = 2n = n + 0 + n(1).$$

Comparing characters, all but one irreducible type occurs in π and each with multiplicity 1. Thus, if $n = 2l + 1$,

$$(l + 1)|G| = (l + 1)2n = (2l + 2)n = (n + 1)n = n^2 + n(1^2).$$

When n is even, the identity has n fixed points and each rotation is fixed point free. For reflections, half (type cr^{2k}) have 2 fixed points, the others have none. Thus

$$|G| = 2n = n + 0 + n/2(2) + n/2(0).$$

Comparing characters, all but two irreducible types occur in π and then with multiplicity 1. Thus, if $n = 2l$,

$$(l + 2)|G| = (l + 2)2n = (2l + 4)n = n^2 + 2n = n^2 + 0 + n/2(2^2) + n/2(0).$$

9. We have seen that $P_{triv} = \frac{1}{n} M_1$, where M_1 is the $n \times n$ matrix consisting of all ones, and $P_{n-1} = I - P_{triv}$.

Each $\pi(g)$ is a permutation matrix; the (i,j)-th entry is 1 if $g\cdot j = i$ and 0 otherwise. For the trivial type, fixing i and j,

$$\frac{1}{n} = \frac{1}{n!}\sum_{g\cdot j=i} 1, \qquad \text{or} \qquad \#\{g\cdot j = i\} = (n-1)!$$

Two cases for the irreducible representation of dimension $n-1$: on and off the diagonal, with $i \neq j$,

$$\frac{n-1}{n} = (n-1)\frac{1}{n!}\sum_{g\cdot i=i}(f(g)-1) \qquad \text{or} \qquad (n-1)!2 = \sum_{g\cdot i=i} f(g),$$

$$\frac{-1}{n} = (n-1)\frac{1}{n!}\sum_{g\cdot j=i}(f(g)-1) \qquad \text{or} \qquad (n-2)!(n-2) = \sum_{g\cdot j=i} f(g),$$

Worth noting: Neither the sgn or $sgn\otimes\pi_{n-1}$ types occur, so, for any i,j,

$$0 = \sum_{g\cdot j=i} sgn(g) \qquad \text{and} \qquad 0 = \sum_{g\cdot j=i} sgn(g)f(g).$$

10. (a) We have that $\sigma(L(g_1)f) = \sigma(g_1)\sigma(f)$ and $\sigma(R(g_2)f) = \sigma(f)\sigma(g_2^{-1})$. Thus

$$\sigma((L\otimes R)(g_1,g_2)f) = \sigma(g_1)\sigma(f)\sigma(g_2^{-1}) = \tilde{\pi}(g_1,g_2)\sigma(f).$$

(b) $(\tilde{\sigma}, Hom_{\mathbb{C}}(V_\sigma, V_\sigma))$ is equivalent to the outer tensor product $(\sigma^*\otimes\sigma, V^*\otimes V)$ as a representation of $G\times G$. Since σ and σ^* are irreducible as representations of G, so is the outer tensor product as a representation of $G\times G$ by Problem 11.

(c) The Hermitian inner product property is straightforward; if we choose an orthonormal basis for V_σ, then the HIP is the usual inner product on $\mathbb{C}^{n^2}$. For invariance under $\tilde{\sigma}$, we invoke unitarity of σ,

$$\begin{aligned}
\langle\tilde{\sigma}(g_1,g_2)T_1, \tilde{\sigma}(g_1,g_2)T_2\rangle &= Trace(\sigma(g_1)T_1\sigma(g_2)^{-1}(\sigma(g_1)T_2\sigma(g_2)^{-1})^*) \\
&= Trace(\sigma(g_1)T_1\sigma(g_2)^{-1}(\sigma(g_2)^{-1})^*T_2^*\sigma(g_1)^*) \\
&= Trace(\sigma(g_1)T_1T_2^*\sigma(g_1)^{-1}) \\
&= Trace(T_1T_2^*) = \langle T_1, T_2\rangle.
\end{aligned}$$

(d) By Schur's Lemma, each $\sigma(\cdot)$ is either 0 or a nonzero multiple of the identity on each M_σ. Thus it is enough to test each irreducible character in the formula since χ_σ is in M_σ. Since $\chi_\sigma^* = \chi_\sigma$ and $d_\sigma\chi_\sigma * \chi_\sigma = \chi_\sigma$,

$$\begin{aligned} d_{\sigma'} Trace(\sigma'(\chi_\sigma)\sigma'(\chi_\sigma)^*) &= d_{\sigma'} Trace(\sigma'(\chi_\sigma * \chi_\sigma^*)) \\ &= Trace(\sigma'(\chi_\sigma)) \\ &= \frac{1}{|G|}\sum_g \chi_{\sigma'}(g)\chi_\sigma(g) \\ &= \begin{cases} 1 & \sigma' \cong \overline{\sigma} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Thus the right-hand side of the formula coincides with the left-hand side on each M_σ. The M_σ-spaces are also mutually orthogonal with respect to both inner products.

11. (a) Since our aim is a heuristic approach to tensors, we opt for an intuitive explanation using coordinates.

We show that $\pi \otimes \pi'$ is irreducible if π and π' are. Recall that

$$V \otimes V' \cong (V^*)^* \otimes V' \cong Hom_{\mathbb{C}}(V^*, V').$$

Choosing orthonormal bases for V^* and V', we may assume that we are working on a vector space W of matrices and π^* and π' act by unitary matrices as follows:

$$(\pi \otimes \pi')(g_1, g_2)M = \pi'(g_2)M\pi^*(g_1^{-1}).$$

We show that any linear transformation $A : W \to W$ that commutes with $\pi^* \otimes \pi'$ is a multiple of the identity.

We note that any linear transformation $A : W \to W$ is of the form

$$A_{B,C}M = BMC,$$

where B and C are square matrices of the correct dimension; if M is $m \times n$, then B is $m \times m$ and C is $n \times n$. Since W has dimension mn, $Hom_{\mathbb{C}}(W, W)$ has dimension m^2n^2. One sees that each such A is linear, and the span of all $A_{B,C}$ has dimension m^2n^2.

Choose B, C nonzero and suppose $A_{B,C}$ commutes with each $\pi \otimes \pi'(g_1, g_2)$. Then

$$B\pi'(g_2)M\pi^*(g_1^{-1})C = \pi'(g_2)BMC\pi^*(g_1^{-1}).$$

Set $g_1 = e$. Then

$$B\pi'(g_2)MC = \pi'(g_2)BMC.$$

Since C is nonzero, we may choose M such that some column of MC is nonzero. Then B and π^* commute, and Schur's Lemma implies B is a multiple of the identity. A similar argument implies that C is a multiple of the identity. Hence $A_{B,C}$ is a multiple of the identity, and the converse of Schur's Lemma implies $\pi \otimes \pi'$ is irreducible.

We leave the other direction for an advanced class on tensors. The proof of the Schur Orthogonality Relations requires only the above direction.

(b) One sees immediately that W is a nonzero subrepresentation of $M(n,\mathbb{C})$ under the induced action by $G \times G$. By part (a), $M(n,\mathbb{C})$ is irreducible, and equality holds.

(c) Note that $\pi(e) = I$ and $\pi(c) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, so the span contains all diagonal matrices. Next note that

$$\pi(r) - cos(2\pi/n)I = sin(2\pi/n)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \pi(r) - \pi(cr) = 2\begin{bmatrix} 0 & 0 \\ sin(2\pi/n) & cos(2\pi/n) \end{bmatrix}.$$

Thus we are able to obtain all of $M(2,\mathbb{C})$ using linear combinations of $\pi(g)$.