INTRO TO GROUP THEORY - APR. 11, 2012
PROBLEM SET 10
GT15/16. GROUP ACTIONS/ CAYLEY'S THEOREM

A group action $\pi : G \to Bij(X)$ is called *faithful* if $\pi(g)x = x$ for all x in X implies $g = e$.

1. (a) Center an equilateral triangle centered at the origin in $\mathbb{R}^2$ with one vertex on the positive x-axis. Consider the usual action of $G = S_3$ on the triangle extended to the plane. For each point x in the plane, find the orbit and stabilizer subgroup.

(b) Repeat (a) with a regular hexagon and $G = D_{12}$.

2. (a) Let $X = \mathbb{R}$. Find the orbits and stabilizers for each x in $\mathbb{R}$ when $G = \{\pm 1\}, \{g > 0\}$, and $\mathbb{R}^*$ act by multiplication. Which actions are transitive? Faithful?

(b) Repeat (a) with $X = \mathbb{C}$ and $G = \{\pm 1\}, \{g \text{ real } > 0\}, \mathbb{R}^*, S^1$, and $\mathbb{C}^*$.

3. Consider the extended complex plane $X = \mathbb{C}_{ext} = \mathbb{C} \cup \{\infty\}$ and the upper half-plane $H = \{x + yi \mid y > 0\}$.

(a) Show that $GL(2, \mathbb{C})$ acts on X by $\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = (az + b)/(cz + d)$. Transitive? Faithful?

(b) Describe the symmetries of X associated to $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(c) Show that if g in $GL(2, \mathbb{C})$ fixes three or more points in X then $g = \pm I$.

(d) Show that $SL(2, \mathbb{R})$ acts on H. That is, show that $Im(z') > 0$ if $Im(z) > 0$ and $z' = \pi(g)z$ for g in $SL(2, \mathbb{R})$. Transitive? Faithful?

4. (a) Consider the natural action of $G = GL(2, \mathbb{R})$ on $X = \mathbb{R}^2$ by linear transformations. Find the orbits and stabilizers of each point in the plane.

(b) Let $SO(3)$ be the subset of $GL(3, \mathbb{R})$ consisting of matrices A such that $AA^T = I$ and $det(A) = 1$. Show that $SO(3)$ is a group under matrix multiplication. (As linear transformations, these matrices represent rotations about a fixed axis through the origin.)

(c) Repeat (a) with $G = SO(3)$ on $X = \mathbb{R}^3$.

(d) If p is prime, adapt (a) to $G = GL(2, \mathbb{Z}/p)$ and $X = (\mathbb{Z}/p)^2$ to compute $|G|$.

Date: July 27, 2012.

5. (a) Explain how $SL(3, \mathbb{Z}/2)$ may be realized as a subgroup of S_7.

(b) Find four generating elements of order 2 in $SL(3, \mathbb{Z}/2)$. (Hint: elementary row operations)

6. Suppose G acts on the set X. If x and y belong to the same orbit, show that $Stab_G(x)$ and $Stab_G(y)$ are conjugate.

7. Let $X = \mathbb{Z}/n$, and define G to be the symmetry group of affine motions of X. That is,

$$G_n = \{\pi_{a,b} \mid a \in (\mathbb{Z}/n)^*,\ b \in \mathbb{Z}/n\}$$

with action on X by

$$\pi_{a,b}(x) = ax + b.$$

(a) Verify that G_n is a group isomorphic to the matrix group $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ for the same range of values for a and b.

(b) Show that D_{2n} is a subgroup of G_n, and that $G_n \cong Aut(D_{2n})$.

(c) For the non-trivial semidirect product G with 21 elements in Problem Set 9, identify G with a subgroup of G_7, and show that $Aut(G) \cong G_7$ by conjugation.

(d) Repeat for a non-abelian semidirect product G with pq elements in Problem Set 9. Show that all such G are isomorphic.

8. (a) Find the order of each element of S_n in terms of its cycle structure, assuming each permutation can be written as a product of disjoint cycles.

(b) Find the smallest m such that $\mathbb{Z}/n$ is a subgroup of S_m.

9. (a) Using the Corollary to Cayley's Theorem, show that a simple group with 60 elements has no subgroups of order 15, 20, or 30.

(b) Using the Corollary to Cayley's Theorem, show that a simple group with 168 elements has no subgroups of order 28, 42, 56, or 84.

10. Describe all groups with exactly two or three subgroups.