INTRO TO GROUP THEORY - MAY 2, 2012
PROBLEM SET 13
GT21/22. INTERNAL PRODUCTS/ FINITE ABELIAN GROUPS

1. Classify all groups of orders 12, 15, 20, 30, and 63.

2. Show that every group of order 12 may be expressed as a non-trivial semidirect product of two abelian groups.

3. Classify all groups of order pq where p and q are distinct primes. Find $Z(G)$.

4. Classify all groups of orders 18, 24, and 28.

5. A finite abelian group has orders of elements (number):

$$12(24), \quad 6(6), \quad 4(12), \quad 3(2), \quad 2(3), \quad 1(1).$$

Find its isomorphism class.

6. Find the isomorphism classes for all abelian groups of orders 16, 200, and 360.

7. Find the isomorphism class of $Aut(\mathbb{Z}/n)$ for $n = 48, 72, 100$. Assume that $(\mathbb{Z}/p^k)^*$ is cyclic if $p \neq 2$ is prime.

8. Suppose $Aut(\mathbb{Z}/n) \cong \mathbb{Z}/6$. Find all possible n. How about $\mathbb{Z}/6 \times \mathbb{Z}/6$? $\mathbb{Z}/6 \times \mathbb{Z}/6 \times \mathbb{Z}/6$?

9. (a) Find the sum of all elements of G is $G = \mathbb{Z}/6, \quad \mathbb{Z}/7, \quad \mathbb{Z}/2 \times \mathbb{Z}/2, \quad \mathbb{Z}/4 \times \mathbb{Z}/2$.

(b) If G is a finite abelian group of odd order, show that $\sum_{x \in G} x = 0$.

(c) What if G is a finite abelian group of even order?

With a little more work, we can describe $Aut(\mathbb{Z}/n) \cong (\mathbb{Z}/n)^*$ completely in terms of FTFAG. We've also looked at $Aut(\mathbb{Z}/2 \times \mathbb{Z}/2) \cong SL(2, \mathbb{Z}/2)$ and $Aut(\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2) \cong SL(3, \mathbb{Z}/2)$. Here's a few more automorphism groups of abelian groups.

10. Find the isomorphism class of $Aut(\mathbb{Z}/4 \times \mathbb{Z}/2)$.

11. Let $H = \mathbb{Z}/4 \times \mathbb{Z}/4$ and $G = Aut(H)$.

Date: May 5, 2012.

(a) Show that each element of G corresponds to an invertible $\mathbb{Z}/4$-linear map from H to H. Explain why $G \cong GL(2, \mathbb{Z}/4)$, the set of 2×2 matrices with entries in $\mathbb{Z}/4$ with determinant 1 or 3.

(b) Show that G has 96 elements.

(c) To verify the Class Equation, first break the normal subgroup $K = SL(2, \mathbb{Z}/4)$ into conjugacy classes of G. This can be done using characteristic polynomials and diagonal matrices, companion matrices, and Jordan forms. (Hint: $48 = (1 + 1 + 6 + 6 + 6) + 8 + 8 + 12$, and use the center.)

(d) To find the other classes, consider characteristic polynomials of the form $p_A(x) = x^2 + kx + 3$ with k in $\mathbb{Z}/4$. (Hint: (2+6+12) + 8 + 8 + 12)

(e) Apply Sylow Theory to G and H.

(f) Find the isomorphism class of $Inn(H)$.

12. Find two non-isomorphic finite groups with equal orders of elements.