INTRO TO GROUP THEORY - FEB. 15, 2012
PROBLEM SET 2 - GT2. DEFINITION OF SUBGROUP

1. We noted that if a group element x has finite order k, then $x^{-1} = x^{k-1}$. Find the order of the following group elements and verify directly.

   (a) (1234) in S_4,

   (b) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in $GL(2, \mathbb{R})$, and

   (c) $\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ in $GL(2, \mathbb{Z}/3)$.

2. For $n \geq 0$, show that $n\mathbb{Z}$ is a cyclic subgroup of $\mathbb{Z}$. Show that all subgroups of $\mathbb{Z}$ are of this form. (Hint: for any positive integers m, n, there exist integers x, y such that $xm + yn = gcd(m, n)$.)

3. (a) Find all subgroups of $\mathbb{Z}/5$, $\mathbb{Z}/7$, and $\mathbb{Z}/12$. (Hint: note that $\langle g \rangle = \langle g^{-1} \rangle$.)
   (b) Find all subgroups of a cyclic group.

4. Find all (cyclic) subgroups of $(\mathbb{Z}/7)^*$ and $(\mathbb{Z}/11)^*$.

5. In the alternating group A_4, compute $(123)(124)$, and $[(12)(34)](123)$. Verify that $H = \{e, (12)(34), (13)(24), (14)(23)\}$ is an abelian subgroup.

6. Count the number of rigid motions for regular dodecahedrons, icosahedrons, octahedrons, and cubes.

7. (a) Show that $S_3 = \langle (12), (123) \rangle$ and $S_3 = \langle (12), (23) \rangle$.
   (b) If H_i are subgroups of G, is $\cup H_i$ a subgroup of G?

8. Label a square's vertices 1(upper right) to 4 counter-clockwise. List all elements of D_8, the symmetry group of the square, in cycle notation, and describe each symmetry geometrically. Find all orders and inverses.

9. D_8 has 5 subgroups with two elements and 3 subgroups with four elements. Describe each in cycle notation and geometrically. (Hint: the A_4 problem)

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10. Recall the group $G = GL(2, \mathbb{R})$ consisting of real 2×2 invertible matrices.

(a) Show that $H = GL(2, \mathbb{Z})$ consisting of integral matrices (integers for entries) with integral inverses is a subgroup of G.

(b) What are the possible $det(A)$ if A is in H? Show that this condition guarantees an integral inverse.