INTRO TO GROUP THEORY - FEB. 22, 2012
PROBLEM SET 3 - GT3. COSETS AND LAGRANGE'S THEOREM

1. (a) If H is a subgroup of the group G, show the Relabeling Rule directly: that $xH = yH$ if and only if x is in yH.

(b) Let $H = \{e, (123), (132)\}$ in A_4. Verify directly that $(12)(34)H = (243)H = (143)H$.

2. Let $H_1 = \{e, (123), (132)\}$ and $H_2 = \{e, (12)(34), (13)(24), (14)(23)\}$ be subgroups of the alternating group A_4. Find the left and right cosets for each subgroup and compare.

3. Use Problem 2 to show that there exists no subgroup of order 6 in A_4.

4. In D_8, the symmetry group of the square, let $H_1 = \{e, (13)(24)\}$ and $H_2 = \{e, (13)\}$. Find the left and right cosets for each subgroup and compare.

5. Let $G = \mathbb{R}$ and $H = \mathbb{Z}$. Consider the map $\omega : \mathbb{R} \to S^1 \subset \mathbb{C}$ given by $\omega(x) = exp(2\pi ix)$. Show that ω defines a bijection between the coset space G/H and S^1.

6. (Linear algebra) Let $O(2)$ be the subset of $GL(2, \mathbb{R})$ consisting of real 2×2 matrices A such that $A^{-1} = A^T$. This is equivalent to the condition $AA^T = A^TA = I$.

(a) Show that $det(A) = \pm 1$.

(b) Show that $O(2)$ is a subgroup of $GL(2, \mathbb{R})$.

(c) Let $SO(2)$ be the subset of $O(2)$ consisting of matrices with $det(A) = 1$. Show that $SO(2)$ is a subgroup of $O(2)$ and that each A in $SO(2)$ can be written as $\begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$.

7. (Linear algebra) Let $G = GL(2, \mathbb{R})$ and H the subgroup of invertible upper triangular matrices of the form $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ where $x, z > 0$.

(a) Show that H is a subgroup of G.

(b) Let $G' = GL_+(2, \mathbb{R})$ be the subgroup of G with $det(A) > 0$. Show that every element A in G' factors uniquely as $A = RU$ where U is in H and R is in $SO(2)$. (Hint: Consider A^TA with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and with $A = RU$.)

(c) Consider the map ω that sends each $A = RU$ to R to $cos(\theta) + isin(\theta)$ in $S^1 \subset \mathbb{C}$. Show that ω defines a bijection between the coset space G'/H and $S^1 \subset \mathbb{C}$.

Date: February 16, 2012.

8. Fix a prime p. Let $G = GL(2, \mathbb{Z}/p)$ and H the subgroup of invertible upper triangular matrices of the form $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ where $x, z \neq 0$.

(a) Show that H is a subgroup of G.

(b) Compute the orders of G and H. How many elements are in the coset space G/H? Interpret using linear algebra.

9. Using Lagrange's Theorem, describe all groups of order 2, 3, and 4. Construct the corresponding multiplication tables. (The entries in the table are $x \circ y$, where x in G indexes the row, y in G indexes the column.)

10. Let $k = |G|$, the order of the finite group G. Show that $x^k = e$ for all x in G. Verify for S_3, A_4, D_8, and $\mathbb{Z}/12$.