INTRO TO GROUP THEORY - MAR. 7, 2012
PROBLEM SET 5 - GT5/6. INDEX 2 THEOREM, ETC.

1. The group of rigid motions of the cube has 24 elements. Explain why this group is S_4 and describe the A_4 subgroup.

2. Let p be a prime > 2. Find an example of a subgroup H of index p in G such that H is not normal in G.

3. (a) List all elements of D_{10} (pentagon) and D_{16} (octagon) in cycle notation and describe geometrically.

(b) Show that D_{2n} is generated by two reflections. Can it be generated by any two reflections?

(c) (Linear Algebra) Show that any reflection of $\mathbb{R}^2$ fixing the origin can be written in the form

$$s_v(w) = w - 2\frac{\langle w, v\rangle}{\langle v, v\rangle}v$$

for some v and all w in $\mathbb{R}^2$.

(d) (Linear Algebra) Show that every rotation about the origin in $\mathbb{R}^2$ can be written as a product of two reflections.

4. (a) Verify the computation of $Z(D_{2n})$ using cycle notation. (Hint: Conjugation Rule for permutations)

(b) Verify the computation of $Z(D_{2n})$ using linear algebra.

5. (a) If S is a nonempty subset of G, show that $Z(S)$ is a subgroup of G. (Note: we also use $C(S)$ for the centralizer of S. For subgroups, C is better.)

(b) Let $H = \{e, (123), (132)\}$ in S_4. Find $C(H)$ and $N(H)$ in A_4 and S_4. Also compute when $G = S_5$.

(c) If $H = \{e, x\}$ with $|x| = 2$, show that $Z(x) = N(H)$. Repeat (b) with $x = (12)$ (S_n only) and $x = (12)(34)$.

6. Using FTFAG, find all abelian groups of order p, $2p$, and pq if p and q are distinct primes. Explain.

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7. Using FTFAG, find all abelian groups of order 8 and 12. Show that the orders of elements in $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/3$ and $\mathbb{Z}/2 \times \mathbb{Z}/6$ correspond.

8. Describe the commutator subgroups and abelianizations of D_{10} and D_{16} geometrically.

9. Find the commutator subgroups of A_4, S_4, and S_5. Abelianizations?

10. (Linear Algebra) (a) Find the commutator subgroups of $O(2)$ and $SO(2)$. Abelianizations?

(b) Find the commutator subgroups of the groups of invertible upper/lower triangular 2×2 matrices. Abelianizations?

(c) Find the commutator subgroups of $SL(2, \mathbb{R})$ and $GL(2, \mathbb{R})$. Abelianizations?