INTRO TO GROUP THEORY - MAR. 14, 2012
PROBLEM SET 6 - GT8/9. GROUP HOMOMORPHISMS AND ISOMORPHISMS

1. (a) Write out the homomorphism $\pi : \mathbb{Z}/6 \to \mathbb{Z}/2 \times \mathbb{Z}/3$ defined by $\pi(1) = (1, 2)$. Note that π is a bijection (and thus an isomorphism).

(b) If m, n are positive integers > 2 such that $gcd(m, n) = 1$, show that

$$\mathbb{Z}/mn \cong \mathbb{Z}/m \times \mathbb{Z}/n.$$

2. Let π be a homomorphism from G to K.

(a) Suppose H is a subgroup of G. Show that

$$\pi(H) = \{\pi(h)|\ h \text{ in } H\}$$

is a subgroup of K. ($\pi(H)$ is called the image of H under π. We also write $\mathrm{Im}(\pi)$ for $\pi(G)$.)

(b) Suppose H is a subgroup of K. Show that

$$\pi^{-1}(H) = \{g \text{ in } G|\ \pi(g) = h \text{ for some } h \text{ in } H\}$$

is a subgroup of G. ($\pi^{-1}(H)$ is called the inverse image of H under π.)

(c) Is the normal condition preserved under image or inverse image?

3. Find all homomorphisms π from $\mathbb{Z}/n$ into S^1. Kernels? How about $\mathbb{Z}$ instead of $\mathbb{Z}/n$?

4. (a) For n in $\mathbb{Z}$ and $z = e^{2\pi i\theta}$ in S^1, show that $\pi : S^1 \to S^1$ given by $\pi(z) = z^n$ is a homomorphism, and find $\mathrm{Ker}(\pi)$. For which n is it a isomorphism?

(b) For s in $\mathbb{R}$ and n in $\{0, 1\}$, show that $\pi : \mathbb{R}^* \to \mathbb{R}^*$ given by $\pi(x) = sgn(x)^n|x|^s$ is a homomorphism, and find $\mathrm{Ker}(\pi)$. For which parameters is it an isomorphism?
(Note: $sgn(x) = x/|x|$.)

(c) For s in $\mathbb{R}$, show that $\pi : \mathbb{C}^* \to \mathbb{R}^*$ given by $\pi(z) = |z|^s$ is a homomorphism, and find $\mathrm{Ker}(\pi)$.

(d) For s in $\mathbb{R}$ and n in $\{0, 1\}$, show that $\pi : GL(2, \mathbb{R}) \to \mathbb{R}^*$ given by

$$\pi(A) = sgn(det(A))^n|det(A)|^s$$

is a homomorphism, and find $\mathrm{Ker}(\pi)$.

5. Find $\mathrm{Im}(\pi)$ for the homomorphisms in Problems 3 and 4.

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6. Prove the Isomorphism Theorems:

(1) If $\pi : G \to H$ is a homomorphism, then $\text{Im}(\pi) \cong \text{G}/\text{Ker}(\pi)$,
(2) If H is a subgroup of G and $N \lhd G$, then $H/(H \cap N) \cong HN/N$, and
(3) If $H, K \lhd G$ and $K \lhd H$, then $H/K \lhd G/K$ and $(G/K)/(H/K) \cong G/H$.

7. For the following G, find all onto homomorphisms $\pi : G \to H$ where H is abelian. Also find $\text{Ker}(\pi)$ in each case. (Hint: check normal subgroups; better: check commutators.)

(1) $G = A_4$,
(2) $G = S_4$,
(3) $G = D_{2n}$, and
(4) $G = SL(2, \mathbb{R})$.

8. (a) If G is cyclic and $\pi : G \to H$ is an isomorphism, show that H is cyclic and π carries any generator to another generator.

(b) For the multiplicative groups $(\mathbb{Z}/5)^*$ and $(\mathbb{Z}/7)^*$, find isomorphisms to some $\mathbb{Z}/n$ under addition.

9. (a) Find an isomorphism between the symmetry groups of rigid motions of the icosahedron and dodecahedron.

(b) Find and isomorphism between the symmetry groups of rigid motions of the cube and the octahedron.

10. Fix $n \geq 3$. Consider the matrix group G with elements of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ where b is in $\mathbb{Z}/n$ and $a = \pm 1$ in $\mathbb{Z}/n$. Show that G is isomorphic to D_{2n}, the dihedral group with $2n$ elements. (Hint: generators and relations)