INTRO TO GROUP THEORY - MAR. 21, 2012
PROBLEM SET 7 - GT10/11. EXAMPLES OF NON-ISOMORPHIC GROUPS/AUTOMORPHISMS

1. Consider the real numbers $\mathbb{R}$ with multiplication $x \circ y = x + y + 2$. Show that $\mathbb{R}$ is a group using $\circ$, and find an isomorphism of $(\mathbb{R}, \circ)$ with $(\mathbb{R}, +)$.

2. Suppose $\pi : G \to K$ is an isomorphism.

(a) If H is a subgroup of G, show that $\pi(H)$ is a subgroup of K, and $\pi(H) \cong H$. If finite, show that $[G : H] = [K : \pi(H)]$. (Recall that $[G : H]$ is the number of cosets of H modulo G.)

(b) If $N \triangleleft G$, show that $\pi(N) \triangleleft K$, and that $G/N \cong K/\pi(N)$. Explain why $|xN| = |\pi(x)\pi(N)|$ as group elements.

3. (a) Find the isomorphism classes of $D_8/Z(D_8)$, $D_{12}/Z(D_{12})$, and $D_{12}/\{e, r^2, r^4\}$.

(b) Let $H = \{e, (12)(34), (13)(24), (14)(23)\}$. Find the isomorphism class of S_4/H.

4. Find the orders of all elements in $G = S_3 \times \mathbb{Z}/2$. What familiar group is G isomorphic to? Construct an explicit isomorphism.

5. (a) If $x^2 = e$ for all x in G and $|G| < \infty$, show that $G \cong \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2$ using induction.

(b) Prove it using linear algebra.

6. If $|G| < \infty$, explain why $|Aut(G)| \leq (|G| - 1)!$. Show that we have equality when $G \cong \mathbb{Z}/p$ for $p = 2, 3$. (More on this problem later.)

7. Calculate $|Aut(G)|$ for $G = \mathbb{Z}/2 \times \mathbb{Z}/4$.

8. Consider the group of matrices Q generated by $M_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in $GL(2, \mathbb{C})$.

(a) Find all elements of Q and their orders. Is Q a familiar group?

(b) Find the isomorphism class of $Inn(Q) \cong Q/Z(Q)$.

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9. (a) Let p be a prime, and recall that $Aut(\mathbb{Z}/p \times \mathbb{Z}/p)$ is isomorphic to $G = GL(2, \mathbb{Z}/p)$. We've seen that $GL(2, \mathbb{Z}/p)$ has $(p^2 - 1)(p^2 - p)$ elements. Find the number of elements in $H = SL(2, \mathbb{Z}/p)$. How many if $p = 3, 5, 7$?

(b) Assuming elements of $Z(G)$ are of the form cI, find $|Z(G)|$ and $|Inn(G)|$. How about $|Z(H)|$ and $|Inn(H)|$?

10. Calculate $|Aut(\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2)|$. Find elements of orders 2, 3, and 7. (Hint: Companion matrices for order 7)