INTRO TO GROUP THEORY - APR. 4, 2012
PROBLEM SET 9
GT13/14. ORDER 8/ SEMIDIRECT PRODUCTS

1. Find the isomorphism classes of the following groups with eight elements:

   (1) $(\mathbb{Z}/16)^*$
   (2) $(\mathbb{Z}/24)^*$
   (3) the power set $P(\{1,2,3\})$ with multipication $\oplus$ (symmetric difference)
   (4) $Z_{S_4}((12)(34))$, the centralizer of $(12)(34)$ in S_4
   (5) $\langle(12)(34),(1234)\rangle$
   (6) $\langle(1234)(5678),(1537)(2648)\rangle$
   (7) $Inn(D_{16})$
   (8) $Out(D_{16})$
   (9) $Aut(D_8)$
   (10) the set of upper triangular matrices with a,b,c in $\mathbb{Z}/2$: $\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}$

2. (a) Find all semidirect products of $\mathbb{Z}/3$ and $\mathbb{Z}/5$.

   (b) Find all semidirect products of $\mathbb{Z}/3$ and $\mathbb{Z}/4$.

   (c) Find all semidirect products of $\mathbb{Z}/3$ and $\mathbb{Z}/2 \times \mathbb{Z}/2$.

   (d) Find all semidirect products of $\mathbb{Z}/3$ and $\mathbb{Z}/7$.

3. Show that $Aut(Q) \cong S_4$. Interpret the elements in $Out(Q)$.

4. Consider the group G defined by the generators and relations

$$y^3 = e, \ x^7 = e, \quad \text{and} \quad yxy^{-1} = x^2.$$

   (a) Show that G is isomorphic to a semidirect product of $\mathbb{Z}/3$ and $\mathbb{Z}/7$.

   (b) Show that there are six elements of order 7 (x^k) and fourteen elements of order 3 (yx^k, y^2x^k).

   (c) Find all automorphisms of G.

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5. (a) If p and q are distinct primes with $p < q$, find all semidirect products of $\mathbb{Z}/p$ and $\mathbb{Z}/q$. What condition is necessary for a non-abelian product?

Now suppose the non-abelian condition in (a) holds. Fix l to be an element of order p in $(\mathbb{Z}/q)^*$. Consider the group G defined by the generators and relations

$$y^p = e, \; x^q = e, \quad \text{and } yxy^{-1} = x^l.$$

(b) Show that G is isomorphic to a semidirect product of $\mathbb{Z}/p$ and $\mathbb{Z}/q$.

(c) Show that there are $q - 1$ elements of order q (x^k) and $q(p-1)$ elements of order p ($y^i x^j, 0 < i \leq p-1$).

(d) Find all automorphisms of G.

The following set of problems concern the quaternion algebra $\mathbb{H}$. This real algebra is associative, distributive, and may be considered as an extension of the complex numbers using the quaternion group Q. As a set,

$$\mathbb{H} = \{x + yi + zj + wk \mid x, y, z, w \in \mathbb{R}\}.$$

The basic operations generalize those for the complex numbers: we add by matching labels, and multiplication is extended in the natural manner from the quaternion group Q :

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

We use the natural distributive property for these operations. In addition, if

$$\alpha = x + yi + zj + wk,$$

then the conjugate of α is

$$\overline{\alpha} = x - yi - zj - wk.$$

One defines the real part of α as

$$Re(\alpha) = x = (\alpha + \overline{\alpha})/2$$

and the imaginary part of α as

$$Im(\alpha) = yi + zj + wk = (\alpha - \overline{\alpha})/2.$$

Finally we define a norm on $\mathbb{H}$ by

$$N(\alpha) = \alpha\overline{\alpha} = \overline{\alpha}\alpha = x^2 + y^2 + z^2 + w^2 \in \mathbb{R}.$$

Thus the norm coincides with the usual length squared for vectors in $\mathbb{R}^4$, and $N(\alpha) = 0$ if and only if $\alpha = 0$.

6. (a) Show that quaternionic multiplication is associative. (Hint: first show for all possible products of i, j, and k.)

(b) Show that $\overline{\alpha\beta} = \overline{\beta}\overline{\alpha}$.

(c) Show that the norm is multiplicative; that is, $N(\alpha\beta) = N(\alpha)N(\beta)$.

(d) Show that if $\alpha \neq 0$, then α is a unit. That is, show that there exists an α^{-1} such that $\alpha\alpha^{-1} = 1$.

(e) Show that $\mathbb{H}^* = \mathbb{H}\backslash\{0\}$ is a non-abelian group under quaternionic multiplication.

(f) Let $U = \{u \in \mathbb{H} \mid N(u) = 1\}$. Show that U is a group under quaternionic multiplication. What familiar space is this? Is it a commutative group?

(g) Find the centers of $\mathbb{H}$ and U.

7. Let $\alpha = 1 - 2k, \beta = 3j + 4k$, and $\gamma = 1 + j$.

(a) Verify that $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

(b) Verify that $\overline{\alpha\beta} = \overline{\beta}\overline{\alpha} \neq \overline{\alpha}\overline{\beta}$.

(c) Verify that $(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1} \neq \alpha^{-1}\beta^{-1}$

(d) Verify that $N(\alpha\beta) = N(\alpha)N(\beta) = N(\beta\alpha)$.

8. We identify the cross product on $\mathbb{R}^3$ with quaternionic multiplication on $Im(\mathbb{H})$. If we identify $v_1 = (y_1, z_1, w_1)$ and $v_2 = (y_2, z_2, w_2)$ with

$$\alpha = y_1 i + z_1 j + w_1 k \quad \text{and} \quad \beta = y_2 i + z_2 j + w_2 k,$$

show that

$$v_1 \times v_2 \quad \text{corresponds to} \quad Im(\alpha\beta) = (\alpha\beta - \beta\alpha)/2.$$

9. For any α in $\mathbb{H}$, we define a map $T_\alpha : \mathbb{H} \to \mathbb{H}$ by

$$T_\alpha(\beta) = \alpha\beta.$$

(a) Show that T_α is $\mathbb{R}$-linear, and find the associated matrix A_α for T_α with respect to the basis $\{1, i, j, k\}$.

(b) Note that $\mathbb{H}$ is a complex vector space with basis $\{1, j\}$ using right multiplication by complex scalars $z = x + yi$. Show that T_α is complex linear with scalar multiplication on the right, and find the matrix B_α with respect to this basis.
(We can multiply by complex numbers on both sides of $\mathbb{H}$, but must fix a side to discuss complex linear maps.)

(c) Show that $\alpha \mapsto A_\alpha \in M(4, \mathbb{R})$ is a one-one homomorphism of associative $\mathbb{R}$-algebras.

(d) Show that $\alpha \mapsto B_\alpha \in M(2, \mathbb{C})$ is a one-one homomorphism of associative $\mathbb{C}$-algebras.

(e) Compute A_α^T, B_α^*, $A_\alpha^T A_\alpha$, and $B_\alpha^* B_\alpha$.

(f) Let $SU(2)$ be the set of 2×2 complex matrices such that $BB^* = I$ and $det(B) = I$. Show that $SU(2)$ is a group and $U \cong SU(2)$.