# INTRO TO GROUP THEORY - FEB. 15, 2012
# SOLUTION SET 1 - GT1. DEFINITION OF A GROUP

1. Group?

(1) $\mathbb{Z}$ under multipication is not a group. Not closed under inverses. Restrict to $\{\pm 1\}$.
(2) $\mathbb{Z}/6$ under multiplication is not a group. 0, 2, 3, 4, 6 have no inverses. Restrict to $\{1, 5\} = \{\pm 1\}$.
(3) the nonzero reals with multiplication $a \circ b = |ab|$ has no identity element. Since $2 \circ 1 = 2$, cancellation says $e = 1$. But $1 \circ -1 = 1 \neq -1$. Restrict to positive reals.

2. Order 1: 0; order 2: 6 (self-inverse); order 3: 4, 8; order 4: 3, 9; order 6: 2, 10; order 12: 1, 11: 5, 7.

3. (a) $(12)(124)(12) = (142) : 1 \to 2 \to 4 \to 4; 2 \to 1 \to 2 \to 1; 3 \to 3 \to 3 \to 3; 4 \to 4 \to 1 \to 2$.

(b) $(124)(13)(142) = (23) : 1 \to 4 \to 4 \to 1; 2 \to 1 \to 3 \to 3; 3 \to 3 \to 1 \to 2; 4 \to 2 \to 2 \to 4$.

(c) $(14)(13)(12) = (1234) : 1 \to 2 \to 2 \to 2; 2 \to 1 \to 3 \to 3; 3 \to 3 \to 1 \to 4; 4 \to 4 \to 4 \to 1$.

4. 5! = 120. $\sigma = (1\ 2)(3\ 4\ 5)$. $\sigma^2 = (3\ 5\ 4)$. $\sigma^3 = (12)$, $\sigma^4 = (3\ 4\ 5)$, $\sigma^5 = (1\ 2)(3\ 5\ 4)$, $\sigma^6 =$ e.

5. p=5: 1 = identity is self inverse with order 1. $4^2 = 16 = 1$, so 4 = -1 is self inverse with order 2. $2 \times 3 = 6 = 1$ so 2 and 3 are inverse to one another. $2^2 = 4, 2^3 = 8 = 3, 2^4 = 1$, so order 4. Same for 3.

p=7: 1 = identity is self inverse with order 1. $6^2 = 36 = 1$, so 6 = -1 is self inverse with order 2. $3 \times 5 = 15 = 1$, so 3 and 5 are inverse to one another. $2 \times 4 = 8 = 1$, so 2 and 4 are inverse to one another. 3, 4, 5, 8 have order 6.

6. $(12)y(123) = (13)$. $\to (12)(12)y(123) = (12)(13) \to y(123) = (132) \to y(123)(132) = (132)(132) \to y = (123)$.

7. Points on the unit circle have Cartesian coordinates $(cos(\theta), sin(\theta))$, which represents the complex number $cos(\theta) + isin(\theta)$. Here $\theta$ is the angle measured from the positive real axis counter-clockwise. Closed under multiplication: $|zw| = |z||w|$, so if $|z| = |w| = 1$, then

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$|zw| = 1$. Specifically, if $z = cos(\theta) + isin(\theta)$ and $w = cos(\omega) + isin(\omega)$, then
$zw = [cos(\theta)cos(\omega) - sin(\theta)sin(\omega)] + i[sin(\theta)cos(\omega) + cos(\theta)sin(\omega)] = cos(\theta+\omega) + isin(\theta+\omega)$.
So when we multiply points on the unit circle, we add the angles. This follows immediately also from Euler's Formula: $exp(i\theta) = cos(\theta) + isin(\theta)$.

8. (a) $det(A)det(B) = (ad - bc)(xw - yz)$. This equals

$$det(AB) = det\begin{pmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{pmatrix} = (ax+bz)(cy+dw) - (ay+bw)(cx+dz).$$

If $det(A) \neq 0, det(B) \neq 0$, then $det(AB) \neq 0$. So $AB$ is in $G$.

(b) $IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+0 & b+0 \\ 0+c & 0+d \end{pmatrix} = A = AI$. det(I) = 1, so I is in G.

(c) $AA^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d/D & -b/D \\ -c/D & a/D \end{pmatrix} = \begin{pmatrix} (ad-cb)/D & (bc-cb)/D \\ (cb-bc)/D & (da-cb)/D \end{pmatrix} = I = A^{-1}A$.

(d) $det(A^{-1}) = (ad - bc)/D^2 = 1/D = [det(A)]^{-1} \neq 0$, so $A^{-1}$ is in $G$. Consider

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This gives equations $ax + bz = 1, ay + bw = 0, cx + dz = 0, cy + dw = 1$. Subtracting $acx + adz = 0$ from $acx + cbz = c$ gives $-adz + bcz = -c$ or $Dz = -c$. Subtracting $acy + bcw = 0$ from $acy + adw = a$ gives $adw - bcw = a$ or $Dw = a$. Substituting into the second equation gives $Dwy + bw = 0$, so either w=0 or $y = -b/D$. Substituting into the third equation gives $-xzD + dz = 0$, so $z = 0$ or $x = d/D$.

(e) Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, and $C = \begin{pmatrix} u & v \\ r & s \end{pmatrix}$.

$$(AB)C = \begin{pmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{pmatrix}\begin{pmatrix} u & v \\ r & s \end{pmatrix} = \begin{pmatrix} axu+bzu+ray+rbw & vax+vbz+say+sbw \\ cxu+dzu+rcy+rdw & vcx+vdz+scy+sdw \end{pmatrix}.$$

On the other hand,

$$A(BC) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} xu+yr & xv+ys \\ zu+wr & zv+ws \end{pmatrix} = \begin{pmatrix} axu+ayr+bzu+bwr & axv+ays+bzv+bws \\ cxu+cyr+dzu+dwr & cxv+cys+dzv+dws \end{pmatrix}.$$

(f) Let $A = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$.

$$AB = \begin{pmatrix} 2 & 4 \\ 6 & 11 \end{pmatrix}, \qquad BA = \begin{pmatrix} 11 & 8 \\ 3 & 2 \end{pmatrix}.$$

$$(AB)^{-1} = \begin{pmatrix} -11/2 & 2 \\ 3 & -1 \end{pmatrix} = B^{-1}A^{-1} = \begin{pmatrix} 1/2 & -3/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$$

$$A^{-1}B^{-1} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1/2 & -3/2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 3/2 & -11/2 \end{pmatrix}$$

9. $det(A) = -7 = 3$. In $(\mathbb{Z}/5)^*$, $3^{-1} = 2$ since $3 \cdot 2 = 1$. So

$$A^{-1} = \begin{pmatrix} 2 \cdot 2 & -3 \cdot 2 \\ -3 \cdot 2 & 1 \cdot 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 2 \end{pmatrix}.$$

$$AA^{-1} = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 4 & 4 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 16 & 10 \\ 20 & 16 \end{pmatrix} = I.$$