INTRO TO GROUP THEORY - APR. 18, 2012
PROBLEM SET 11
GT17/18. SYMMETRIC GROUPS/ CONJUGACY

1. (a) Representative, order, count, centralizer count:

   (1) $e, 1, 1, 120,$
   (2) $(12), 2, 5 \times 4/2 = 10, 2 \times 6 = 12$ (the centralizer is a copy of $\mathbb{Z}/2 \times S_3$),
   (3) $(123), 3, 5 \times 4 \times 3/3 = 20, 3 \times 2 = 6,$
   (4) $(1234), 4, 5 \times 4 \times 3 \times 2/4 = 30, 4,$
   (5) $(12345), 5, 5!/5 = 24, 5,$
   (6) $(123)(45), 6, 5!/6 = 20, 6,$
   (7) $(12)(34), 2, 5!/8 = 15, 8$ (the centralizer is a copy of D_8)

   (b) $1 + 10 + 20 + 30 + 24 + 20 + 15 = 120$

   (c) For the trace formula, you will need that n-th roots of unity will satisfy

$$1 + x + x^2 + \cdots + x^{n-1} = (1 - x^n)/(1 - x) = 0.$$

2. (a) If x is in the center of S_n, then $x(12 \dots n-1)x^{-1} = (12 \dots n-1)$ implies that x fixes the label n. This holds for any $(n-1)$-cycle, so x fixes all labels and $x = e$. Alternatively, if x contains a non-trivial cycle then the centralizer of x has less than $n!$ elements by counting.

   (b) See video for lists.

   (c) $(n - k)!/k$

   (d) $H = \{e, (12)\} \cong \mathbb{Z}/2$, A_n is normal in S_n, and $S_n = A_n \cup (12)A_n$. if x is in A_n, send x to $(0, x)$ and $(12)x$ to $(1, x)$ in $\mathbb{Z}/2 \times A_n$ (set product). The defining homomorphism is given by conjugation by (12).

3. First $[\pi\sigma(e)]x = \sigma(e)x = x$. Then

$$[\pi\sigma(gh)]x = [\pi(\sigma(g)\sigma(h))]x = [(\pi\sigma(g))(\pi\sigma(h))]x = \pi\sigma(g)[\pi\sigma(h)x].$$

Date: August 21, 2012.

(1) $\pi\sigma(e) : I$,
(2) $\pi\sigma(12) : 1 \to 2, 2 \to 1, 3 \to 3$,
(3) $\pi\sigma(13) : 1 \to 1, 2 \to 3, 3 \to 2$,
(4) $\pi\sigma(23) : 1 \to 3, 2 \to 2, 3 \to 1$,
(5) $\pi\sigma(123) : 1 \to 3, 2 \to 1, 3 \to 2$,
(6) $\pi\sigma(132) : 1 \to 2, 2 \to 3, 3 \to 1$.

4. (a) A_5: e, 20 elements of order 3 (three-cycles), 24 elements of order 5 (five-cycles), 15 elements of order 2 (products of disjoint two-cycles).

I_{60} : e, 20 rotations by 120 degrees (10 opposing triangle pairs with 2 non-trivial rotations), 24 rotations about vertices (6 opposing vertex pairs with 4 non-trivial rotations), 15 rotations by 180 degrees (along 15 opposing edge pairs).

(c) The classes are given in (a), except for the 24 elements of order 5. These split into two classes of 12 elements consisting of inverse pairs. As five-cycles, we check if different from (12345) by even or odd relabelings. Geometrically these classes represent rotations by 72 and 144 degrees. Note that the inverse rotates by the same angle in the opposite direction.

(d) Conjugation by elements of S_5 relabels elements of A_5 faithfully, so S_5 is isomorphic to a subgroup of $Aut(A_5)$. Since A_5 has no subgroups of order 15 or 30, A_5 is generated by (123) and (45132). Now $(123)(45123)(132) = (45231)$, so an automorphism of A_5 requires a choice of three-cycle (abc) and a five-cycle of the form $(deabc), (debca)$, or $(decab)$. Thus there are at most $20 \times 2 \times 3 = 120$ automorphisms, and $Aut(A_5) \cong S_5$ by conjugation. Since $Z(A_5) = \{e\}$, $Inn(A_5) \cong A_5$, and $Out(A_5) \cong \mathbb{Z}/2$.

5. (a) Noting the Conjugation/Relabeling Rule, repeated conjugation among these transpositions generate all transpositions, and in turn all products of transpositions yield S_n.

(b) Noting the Conjugation/Relabeling Rule, conjugation of (12) by powers of $(12 \ldots n)$ yields the set in (a).

(c) Automorphisms permute conjugacy classes. If we show that no other class has the same number of elements as the transpositions, then the result follows. Since automorphisms preserve orders of elements, we consider only classes for elements of order 2. These elements are products of disjoint transpositions. By counting the elements in each class, we arrive at the equation for centralizer orders:

$$2(n-2)! = 2^k k!(n-2k)!$$

where k is the number of disjoint two cycles in the target element. If $k > 1$, this simplifies to the equation

$$(n-2)(n-3)\ldots(n-2k+1) = 2^{k-1}k! = k(2k-2)(2k-4)\ldots(4)(2).$$

There are no solutions for n when $k = 2$, and if $k = 3$ then $n = 6$. When $k \geq 4$, the left-hand side of the equation is strictly greater than the right-hand side. To prove this, first show the inequality is true when $n = 2k$ and then still holds when k is fixed and n is increased.

When $n = 6$, the conjugacy classes for (12) and (12)(34)(56) both have 15 elements.

(d) Suppose $n \neq 6$ and consider pairs of transpositions. By (c), automorphisms carry commuting (resp. non-commuting) pairs to similar pairs. Inductively, we see that cycle structure is preserved under automorphisms, and n-cycles are carried to n-cycles.

If $n = 6$, the classes for (123456) and (123)(45) both have 120 elements.

(e) Parts (c) and (d) show that, at most, an automorphism has $n(n-1)/2$ possibilities for where to send (12) and $2(n-2)!$ possibilities for where to send $(12\dots n)$; note the equation $(12)(12\dots n)(12) = (21\dots n)$. Thus there are at most $n!$ automorphisms; since the center of S_n is trivial, $Aut(S_n) = Inn(S_n) \cong S_n$. Note that we have two descriptions of automorphisms: by conjugation (relabeling), or by assigning a transposition and an n-cycle.

(f) Suppose another such subgroup H exists. Then it is normal by the Index 2 Theorem, and $H \cap A_n$ is a normal subgroup of S_n and thus A_n. Since A_n is simple when $n \geq 5$, $H \cap A_n = \{e\}$. Now H contains all but one element of $S_n \backslash A_n$, so H contains a pair of transpositions by counting. The product of these transpositions lies in A_n, a contradiction.

(g) Suppose $n > 6$, and recall part (e). Since A_n is characteristic and S_n acts faithfully by conjugation, $|Aut(A_n)| \geq n!$. Since A_n is generated by the single class of three-cycles, we adapt the argument of part (c) to show that 3-cycles (and also cycle structure) are preserved under automorphisms of A_n. Noting the unique way to multiply a pair of 3-cycles to get another 3-cycle,

$$(bad)(abc) = (bcd) \qquad \text{implies} \qquad \pi(bad)\pi(abc) = \pi(bcd),$$

and we see that these automorphisms send A_4-subgroups fixing $(n-4)$ labels to similar subgroups. Inductively we see that these automorphisms send A_{n-1}-subgroups fixing a single label to similar subgroups. Thus an automorphism preserving 3-cycles permutes these n subgroups, and, by considering the fixed points, the action is faithful. Thus $|Aut(A_n)| = n!$, and $Aut(A_n) \cong S_n$. See (h) for $n = 6$.

(h) (Outline) When $n = 6$, we obtain more automorphisms beyond relabeling. Using (b), (c) and (d), it is enough to consider pairs $x = (ab)(cd)(ef)$ and $y = (xyz)(uv)$ that generate S_6. A necessary condition is that repeated conjugation of x by y produce six distinct elements; compare with (12) and (123456). One sees that this occurs if we have $x = (ab)(cd)(ef)$ and $y = (abc)(de)$; otherwise, $(abc)(ef)$ yields three conjugates of x. This yields $15 \times 48 = 720$ extra automorphisms beyond relabeling, and $Out(S_6) \cong \mathbb{Z}/2$.

Note that outer automorphisms carry elements of the form (abc) to $(xyz)(uvw)$. Five-cycles and disjoint products of two-cycle pairs are preserved. Now A_6 is preserved since characteristic, and the automorphisms permute 12 subgroups isomorphic to A_5 in A_6, with outer automorphisms interchanging the types. Counting now shows $Aut(S_6) \cong Aut(A_6)$.

6. (a) The classes of D_8 are

$$\{e\}, \{(13)(24)\}, \{(14)(23), (12)(34)\}, \{(13), (24)\}, \{(1234), (1432)\},$$

so $8 = 1 + 1 + 2 + 2 + 2$.

Using the usual generators and relations, the classes for D_{10} are

$$\{e\}, \{r, r^4\}, \{r^2, r^3\}, \{c, cr, cr^2, cr^3, cr^4\},$$

so $10 = 1 + 2 + 2 + 5$.

(b) When n is odd, the only nontrivial, proper normal subgroups are those of the rotation subgroup. So $\langle r^3 \rangle$, $\langle r^5 \rangle$, and R_{15}.

When n is even, we have the subgroups of the rotation subgroup and $\langle r^2, c \rangle$, which is a copy of D_n. So we have D_{30}, $\langle r^3 \rangle$, $\langle r^5 \rangle$, $\langle r^{15} \rangle$, and R_{30}.

7. (a) First note that $Z(G(12)) = \{e, y^2\}$. Since $\{e, x, x^2\}$ is a normal subgroup of $G(12)$,, $\{x, x^2\}$ forms a class of elements of order 3. Multiplying by the central element y^2, we obtain the class $\{xy^2, x^2y^2\}$, whose elements have order 6. The conjugacy class of y is $\{y, xy, x^2y\}$ and, again using y^2, the class of y^3 is $\{y^3, xy^3, x^2y^3\}$.

If a normal subgroup contains y or y^3, it contains both classes. Including the identity, the order exceeds 7 and is all of $G(12)$. Thus the non-trivial, proper normal subgroups are $\langle y^2 \rangle$, $\langle x \rangle$, and $\langle xy^2 \rangle$ (index 2).

(b) The center is trivial, but we have the normal subgroup $\langle x \rangle$. This breaks into classes $\{e\}, \{x, x^2, x^4\}$, and $\{x^3, x^5, x^6\}$. The remaining 14 elements of order 3 break into two classes of order 7 by checking the centralizer of $y : \{yx^i\}$ and $\{y^2x^i\}$.

If a normal subgroup contains an element of order 3, consideration of classes implies the subgroup is $G(12)$. Thus the only non-trivial, proper, normal subgroups is $\langle x \rangle$.

(c) The center is trivial, but we have the normal subgroup $\langle x \rangle$. This breaks into classes $\{e\}$ and k classes with p elements. The remaining $pq - q = q(p - 1)$ elements break into $p - 1$ classes of q elements of order p: $\{y^ix^k\}$ indexed by i. Since a normal subgroup containing a non-trivial power of y contains all such powers, the only non-trivial, proper normal subgroups are subgroups of $\langle x \rangle$. Since q is prime, the only possibility is $\langle x \rangle$.

8. (a) Suppose z is in $Z(G)$. Then $z(gxg^{-1}) = g(zx)g^{-1}$. For the group action, $eC_x = C_x$ and $z_1(z_2C_x) = z_1C_{z_2x} = C_{z_1z_2}$.

(1) $Z(Q) = \{\pm 1\}$. Trivial action since nontrivial classes are closed under multiplication by -1.

(2) $Z(D_8) = \{e, (13)(24)\}$. Also trivial since nontrivial classes are closed under multiplication by $(13)(24)$.

(3) $Z(D_{4n}) = \{e, r^n\}$. Effect of r^n : classes of the form $\{r^k, r^{-k}\}$ are sent to $\{r^{n+k}, r^{n-k}\}$. If n is even, the classes $\{cr^{even}\}$ and $\{cr^{odd}\}$ are fixed, but switched if n odd.

(4) $Z(G(12)) = \{e, y^2\}$. Effect of y^2 : switches $\{e\}$ and $\{y^2\}$, switches classes for elements of order 3 and 6, switches classes for elements of order 4.

(b) If π is in $Aut(G)$ then $\pi(gxg^{-1}) = \pi(g)\pi(x)\pi(g)^{-1}$ and $\pi(C_x) = C_{\pi(x)}$. For the group action, $IC_x = C_{Ix} = C_x$ and $\sigma(\pi(C_x)) = \sigma C_{\pi(x)} = C_{\sigma\pi(x)}$. Conjugacy classes are preserved under inner automorphisms since $gC_xg^{-1} = C_{gxg^{-1}} = C_x$. Thus the action by $Aut(G)$ passes to $Out(G)$.

(1) A_5: interchanges the classes for 5-cycles,

(2) S_6 : since an outer automorphism interchanges the classes for (123456) and (123)(45), the classes for (123)(456) and (123) (and (12)(34)(56) and (12)) interchange by raising to powers. Since automorphisms permute Sylow 5-subgroups and their normalizers, the classes for e, (12345), (12)(34), and (1234) remain fixed. A typical normalizer is $\langle(12345), (1243)\rangle$. It follows that the class for (1234)(56) is also fixed.

(3) $G(12)$: first, the center has 2 elements, so $Inn(G)$ has six elements and no element of order 6. So $Inn(G) \cong S_3$. Automorphisms are determined by values on x and y. One shows there are 12 automorphisms, and the non-trivial element of $Out(G)$ is represented by $x \mapsto x$ and $y \mapsto y^3$. This automorphism interchanges the classes for y and y^3.

(4) $G(21)$: $Z(G)$ is trivial, and $|Aut(G)| = 42$, so $Out(G) \cong \mathbb{Z}/2$, with non-trivial element represented by $\pi_{3,0} : y \mapsto y, x \mapsto x^3$. Thus the classes with 7 elements are fixed, and the classes with 3 elements switch.

9. (a) The center has p or more elements. Trial and error shows that central elements have $a = c = 0$.

(b) Since the center has order p, there are $p^3 - p = p(p-1)(p+1)$ non-central elements. Thus there are $p^2 - 1$ non-central conjugacy classes, each with p elements.

10. (a-b) Sylow theory will simplify some tedious work in (b). In all but the first case, the centralizer is generated by the element.; we will see that the centralizer of the first element is isomorphic to D_8. We list by order, number of conjugates, minimal polynomial, type:

(1) 2, 21, $(x+1)^2$, Jordan canonical form,

(2) 4, 42, $(x+1)^3$, Jordan canonical form,

(3) 7, 24, $x^3 + x + 1$, rational canonical form,

(4) $7,\ 24,\ x^3 + x^2 + 1$, rational canonical form,
(5) $3,\ 56,\ x^3 + 1$, rational canonical form,

(c) Normal subgroups are severely restricted: they are disjoint unions of conjugacy classes, they contain the identity, and the order divides the group order. With the counts from (a-b), only $\{e\}$ and G work.