INTRO TO GROUP THEORY - FEB. 15, 2012
SOLUTION SET 2 - GT2. DEFINITION OF SUBGROUP

1. (a) $\sigma = (1234)$, $\sigma^2 = (13)(24)$, $\sigma^3 = (1432), \sigma^4 = e$. Order 4 and $(1234)(1432) = e$.

(b) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, A^2 = -I, A^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, A^4 = I$. Order is 4, and $AA^3 = I$.

$$(c)A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, A^2 = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, A^3 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, A^4 = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, A^5 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, A^6 = I.$$

Order is 6 and $AA^5 = I$.

2. If $n = 0$, then $H = n\mathbb{Z}$ is the identity subgroup. Suppose $n > 0$.

(1) Closed under addition: if x, y are in H, then $x = jn$ and $y = kn$ for j, k in $\mathbb{Z}$. Then $x + y = (j + k)n$ is in $n\mathbb{Z}$.
(2) Closed under Inversion: If x is in H, then $x = jn$ for j in $\mathbb{Z}$. Then $-x = (-j)n$ is in H.
(3) Nonempty: for any n, 0 is in H.
(4) Cyclic: $H = \langle n \rangle$.

Suppose H is a subgroup of $\mathbb{Z}$. If $H = \{0\}$, done. If not, there exists a smallest positive integer m in H. (If not, there exists a negative integer x by assumption; then $-x$ is positive and in H.) Consider the subgroup $K = m\mathbb{Z}$. If $H \neq K$, then there exists a m' in H that is not a multiple of m. Thus $0 < d = gcd(m', m) < m$. Now there exist integers i, j such that $im' + jm = d$. By closure under addition, d is in H and $0 < d < m$, a contradiction.

3. The infinite cyclic case $G = \mathbb{Z}$ is Problem 2. In the finite cyclic case, we consider subgroups generated by single elements $H_g = \langle g \rangle$. $\mathbb{Z}/5 = H_i$ for $0 < i < 4$. Check directly or note that $gcd(i, 5) = 1$. Then there exists x, y in $\mathbb{Z}$ such that $xi + y5 = 1$. Since 1 generates $\mathbb{Z}/5$, $H_i = \mathbb{Z}/5$. Same argument for $\mathbb{Z}/7$ or any $\mathbb{Z}/p$ with p prime.

For $\mathbb{Z}/12$,

(1) $H_1 = H_5 = H_7 = H_{11} = \mathbb{Z}/12$,
(2) $H_2 = H_{10} = \{0, 2, 4, 6, 8, 10\}$,
(3) $H_3 = H_9 = \{0, 3, 6, 9\}$,
(4) $H_4 = H_8 = \{0, 4, 8\}$, and
(5) $H_6 = \{0, 6\}$.

For general $\mathbb{Z}/n$, there will be one exactly one subgroup $H = \langle d \rangle$ with n/d elements for each divisor d of n. One can adapt the above proof for $\mathbb{Z}$ to show no others exist.

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4. Now the operation is multiplication instead of addition. $(\mathbb{Z}/5)^* = \{1,2,3,4\} = H_2 = H_3$. $H_4 = \{1,4\}$. Of course $H_1 = \{1\}$.

When $G = \mathbb{Z}/7$, $H_1 = \{1\}$. Also

(1) $(\mathbb{Z}/7)^* = \{1,2,3,4,5,6\} = H_3 = H_5$,
(2) $H_2 = H_4 = \{1,2,4\}$, and
(3) $H_6 = \{1,6\}$.

5. $(123)(124) = (13)(24) : 1 \to 2 \to 3; 2 \to 4 \to 4; 3 \to 3 \to 1; 4 \to 1 \to 2.$
$[(12)(34)](123) = (243) : 1 \to 2 \to 1; 2 \to 3 \to 4; 3 \to 1 \to 2; 4 \to 4 \to 3.$

To see that $H = \{e, (12)(34), (13)(24), (14)(23)\}$ is an abelian subgroup, we note that

$$[(ab)(cd)][(ac)(bd)] = (ad)(bc) = [(ac)(bd)][(ab)(cd)].$$

6. Dodecahedron: 20 vertices, each surrounded by three pentagons. A rigid motion permutes the vertices, so a given vertex has 20 choices. Once chosen, there are three ways to orient the surrounding pentagons. So 60 rigid motions.

Icosahedron: 12 vertices, each surrounded by five triangles. Same method gives 60 rigid motions.

Octahedron: 6 vertices, each surrounded by four triangles. Same method gives 24 rigid motions.

Cube: 8 vertices, each surrounded by three squares. Same method gives 24 rigid motions.

7. (a) $(123)(123) = (132), (12)(123) = (23), (123)(12) = (13)$. Note that $(12)(23) = (123)$, so $S_3 = \langle (12), (123) \rangle \subseteq \langle (12), (23) \rangle$.

(b) No. $H_1 = \langle (12) \rangle$, $H_2 = \langle (123) \rangle$. $H_1 \cup H_2$ is not a subgroup of S_3.

8. Element, order, inverse, geometry

(1) e, 1, e, no change to square,
(2) (1234), 4, (1432), rotation counter-clockwise by 90 degrees,
(3) (13)(24), 2, self-inverse, rotation by 180 degrees,
(4) (1432), 4, (1234), rotation clockwise by 90 degrees,
(5) (13), 2, self-inverse, reflect across line $y = -x$,
(6) (24), 2, self-inverse, reflect across line $y = x$,
(7) (12)(34), 2, self-inverse, reflect across y-axis,
(8) (14)(23), 2, self-inverse, reflect across x-axis.

9. For subgroups with two elements, $H_x = \{e, x\}$, where x is an element of order 2 in Problem 8's list.

For subgroups with four elements, we have

(1) the rotation subgroup $R = \{e, (1234), (13)(24), (1432)\}$,
(2) $D = \{e, (13), (24), (13)(24)\}$ (generated by diagonal reflections), and
(3) $H = \{e, (14)(23), (12)(34), (13)(24)\}$ (generated by the h/v reflections).

10. (a) The matrix product of two matrices with integer entries has integer entries. Since $(AB)^{-1} = B^{-1}A^{-1}$, H is closed under multiplication. Since $(A^{-1})^{-1} = A$, H is closed under inversion. H is nonempty since $I^{-1} = I$.

(b) Since $det(A^{-1}) = 1/det(A)$ and $det(A)$ and $det(A^{-1})$ are integers,

$$det(A) = det(A^{-1}) = \pm 1.$$

If $det(A) = \pm 1$, then

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ or } \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}.$$