INTRO TO GROUP THEORY - FEB. 29, 2012
SOLUTION SET 4 - GT4. NORMAL SUBGROUPS AND QUOTIENT GROUPS

1. (a) $HH = H$ is straightforward.

(1) $H_1H_2 = \{ee, e(13), (12)e, (12)(13)\} = \{e, (13), (12), (132)\}$,
(2) $H_2H_1 = \{ee, e(12), (13)e, (13)(12)\} = \{e, (12), (13), (123)\}$, and
(3)

$$\begin{aligned} H_1H_3 &= \{ee, e(123), e(132), (12)e, (12)(123), (12)(132)\} \\ &= \{e, (123), (132), (12), (23), (13)\} = H_3H_1 = S_3. \end{aligned}$$

2. (a) Closed under multiplication: if hn and $h'n'$ are in HN, then, for some n'' in N, $hnh'n' = hh'n''n'$ is in HN; that is, $h'nh'^{-1} = n''$.

Closed under inverse: If hn is in HN, then $(hn)^{-1} = n^{-1}h^{-1} = h^{-1}n'$ is in HN; again, $h^{-1}n^{-1}h = n'$.

Nonempty: since e is in both H and N, $e = ee$ is in HN.

(b) If xN and yN are in H/N, then xy is in H and xyN is in H/N. If xN is in H/N then x^{-1} is in H and $(xN)^{-1} = x^{-1}N$ is in H/N. Finally e is in H, so $N = eN$ is in H/N.

Let H be the subset $\cup xN$, where x runs over a set of representative for the cosets H'. We claim H is a subgroup of G. If x and y are in H, then xN and yN are in H'. Thus xyN is in H', so xy is in H. If x is in H, then xN is in H'. But $(xN)^{-1} = x^{-1}N$ is in H', so x^{-1} is in H. Finally $N = eN$ is in H', so e is in H.

(c) Suppose x is in G and h is in H. Now $xNhN(xN)^{-1} = xhx^{-1}N = h'N$ since $H \triangleleft G$.

Let $H = \{e, (12)\}$ and $N = \{e, (13)(24)\}$. Since G/N is abelian, every subgroup of G/N is normal.

3. (a) $G = A_4$, $H = \{e, (12)(34), (13)(24), (14)(23)\}$, and $K = \{e, (12)(34)\}$.

(b) We have seen that the intersection of subgroups H_i is a subgroup. We show the normal property. If x is in $\cap H_i$, then x is in each H_i. If we conjugate by x in G, then xhx^{-1} is in H_i for each i by normality. Thus xhx^{-1} is in $\cap H_i$.

4. Straightforward.

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5. (a) Suppose $g_i h g_i^{-1}$ is in H for a set of generators $\{g_i\}$. Since $g_i^k = e$ for some k, $g_i^{-1} = g_i^{k-1}$, so $g_i^{-1} h g_i$ is in H. Since $(xy)h(xy)^{-1} = x(yhy^{-1})x^{-1}$, the result holds for any product of generators and their inverses.

(b) - (e) Computation. It will be easier if we note the Conjugation Rule: for σ in S_n, $\sigma(ab\dots d)\sigma^{-1} = (\sigma(a)\sigma(b)\dots\sigma(d))$.

6. (a) $(xN)^k = e$ if and only if x^k is in N. Let j be the smallest positive integer such that x^j is in N. Let $H = \langle x\rangle$. Then $H \cap N$ is a cyclic subgroup of H with order $|x|/j$, and $|xN| = j$.

(b) Each non-identity element of A_4/H has order 3 since $|A_4/H| = 3$.. In $D_8/Z(D_8)$, each non-identity element has order 2 since $(1234)^2 = (1432)^2 = (13)(24)$ and $(13)^2 = e$.

7. Since each subgroup of G is cyclic, $G = \langle g\rangle$, $N = \langle g^k\rangle$, and we can choose $0 \le k < |G|$ and k divides $|G|$ if $k \ne 0$. Thus $G/N = \{N, gN, \dots, g^{k-1}N\}$.

We can represent each subgroup of $\mathbb{Z}/n$ in the form $H = \langle d\rangle$ where $0 \le d < n$ and d divides n if $d \ne 0$. So $(\mathbb{Z}/n)/H$ is cyclic with $n/(n/d) = d$ elements.

8. (a) No.

$$\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & -2\end{pmatrix}\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix} = \begin{pmatrix}2 & -1\\ 5 & -2\end{pmatrix}.$$

No.

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

(b) $SO(2)$ is a subgroup since $det(AB) = det(A)det(B)$ and $det(A^{-1}) = 1/det(A)$. Normal follows since $det(gAg^{-1}) = det(gg^{-1}A) = det(A)$. There is a bijection between $SO(2)$ and the elements with $det(A) = -1$ given by

$$R \to \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} R.$$

So there are two cosets: $SO(2)$ and $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} SO(2)$,

(c) Abelian follows from trig identities. Since $det(A) = -1$, A is in the coset $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} SO(2)$. Thus A can be written in the form

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}cos(\theta) & -sin(\theta)\\ sin(\theta) & cos(\theta)\end{pmatrix} = \begin{pmatrix}cos(\theta) & -sin(\theta)\\ -sin(\theta) & -cos(\theta)\end{pmatrix}.$$

Elements of $SO(2)$ are rotations (about the origin by θ counter-clockwise). The remaining elements are reflections; they have eigenvalues ± 1.

(d) Split into two cases based on the sign of $det(A)$, and revise the solution for $det(A) > 0$ by adjusting signs if needed.

9. (a) For the subgroup and normal properties, argue as in Problem 8(b). $SO(2)$ is not normal in $SL(2,\mathbb{R})$ using the same elements as in Problem 8(a).

(b) Show $Trace(AB) = Trace(BA)$. In general, conjugation preserves similarity classes, so characteristic polynomials and coefficients, minimal polynomials, canonical forms, etc.

10. (a) Closed under multiplication: if $|x| = j$ and $|y| = k$ then $(xy)^{jk} = x^{jk}y^{jk} = e$. Closed under inverse: $|x^{-1}| = |x|$. Nonempty: $|e| = 1$.

(b) $|gxg^{-1}| = |x|$ since $(gxg^{-1})^k = gx^kg^{-1} = gg^{-1} = e$. Not a subgroup in general. In $GL(2,\mathbb{Z})$, let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Then $|A| = 4$ and $|B| = 3$, and $AB = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, which has infinite order.

(c) $\mathbb{Q}/\mathbb{Z}$. The former is the set of elements of infinite order in $\mathbb{R}/\mathbb{Z}$; these correspond to cosets for irrationals. A non-identity coset from $\mathbb{R}/\mathbb{Q}$ is of the form $x+\mathbb{Q}$ with x irrational. This quotient group is uncountable since $\mathbb{Q}$ is countable.

(d) $T(\mathbb{R}^*) = \{\pm 1\}$. If z is in $\mathbb{C}$ and $z^k = 1$, then $|z^k| = |z|^k = 1$ and $|z| = 1$. With this, one has $T(\mathbb{C}^*) = \{exp(i\theta) : \theta \in 2\pi i\mathbb{Q}\}$.