INTRO TO GROUP THEORY - MAR. 14, 2012
SOLUTION SET 6 - GT8/9. GROUP HOMOMORPHISMS AND ISOMORPHISMS

1. (a) Closed under multiplication: if $k = \pi(h)$ and $k' = \pi(h')$, then

$$kk' = \pi(h)\pi(h') = \pi(hh').$$

Since hh' is in H, kk' is in $\pi(H)$.

Non-empty: $e_K = \pi(e_H)$ is in $\pi(H)$.

Closed under inverse: if $k = \pi(h)$ then $k^{-1} = \pi(h^{-1})$ is in $\pi(H)$.

(b) Closed under multiplication: if g, g' in $\pi^{-1}(H)$, then $\pi(g) = h$ and $\pi(g') = h'$, and $\pi(gg') = \pi(g)\pi(g') = hh'$. Thus gg' is in $\pi^{-1}(H)$.

Non-empty: $\pi(e_G) = e_H$, so e_G is in $\pi^{-1}(H)$.

Closed under inversion: if $\pi(g) = h$, then $\pi(g^{-1}) = h^{-1}$. Thus g^{-1} is in $\pi^{-1}(H)$.

(c) Image: if $N \triangleleft G$, then $\pi(N) \triangleleft K$ when π is onto, but not in general. Suppose $h = \pi(g)$. Then

$$h\pi(N)h^{-1} = \pi(g)\pi(N)\pi(g^{-1}) = \pi(gNg^{-1}) = \pi(N).$$

If not, consider the inclusion of $G = \{e, (12)\}$ into $K = D_8$, the symmetry group of the square. G is normal in itself, but not in D_8.

Inverse image: $\pi^{-1}(N)$ is always normal if $N \triangleleft K$. If h is in $\pi^{-1}(N)$, then $\pi(h) = n$ and knk^{-1} is in N. If g is in G, then $\pi(ghg^{-1}) = \pi(g)\pi(n)\pi(g)^{-1}$ is in N. Thus ghg^{-1} is in $\pi^{-1}(N)$.

2. (a) The homomorphism is determined by $\pi(1)$ and will preserve order of elements since an isomorphism.

(1) $\pi(1) = (1, 2)$,
(2) $\pi(2) = (0, 4)$,
(3) $\pi(3) = (1, 0)$,
(4) $\pi(4) = (0, 2)$,
(5) $\pi(5) = (1, 4)$, and
(6) $\pi(6) = \pi(0) = (0, 0)$,

(b) Define a homomorphism $\pi : \mathbb{Z} \to \mathbb{Z}/m \times \mathbb{Z}/n$ by $\pi(1) = (1, 1)$.

For the homomorphism property, $\pi(i + j) = (i + j, i + j) = (i, i) + (j, j) = \pi(i) + \pi(j)$.

By 1(a), $\mathrm{Im}(\pi)$ is a subgroup of $\mathbb{Z}/m \times \mathbb{Z}/m$. We count the number of elements in $\mathrm{Im}(\pi)$.

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If k is in $\text{Ker}(\pi)$, then $\pi(k) = (0,0)$ and k is a multiple of both m and n. Since $gcd(m,n) = 1$, k is a multiple of mn. Conversely, any multiple of mn is in $\text{Ker}(\pi)$. So $\text{Ker}(\pi) = mn\mathbb{Z}$.

Since π maps $\mathbb{Z}$ onto $\text{Im}(\pi)$, we have that $\mathbb{Z}/mn\mathbb{Z} \cong \text{Im}(\pi)$. See 6(a) below. Now the quotient group is cyclic with mn elements, so π is onto, and the result follows.

3. If $\pi(1) = z$, then the order of z divides n. That is $z = exp(k2\pi i/n)$ for some integer $0 \leq k < n$. If $d = gcd(k,n)$, and $\text{Ker}(\pi) = \langle n/d \rangle$. Then $\text{Ker}(\pi)$ has d elements and $\text{Im}(\pi)$ has n/d elements. (Note that since $|z|^k = 1$, $|z| = 1$, and z is in S^1 automatically.)

With $\mathbb{Z}$, if $\text{Ker}(\pi) = n\mathbb{Z}$, then we are in the first case. Otherwise $\text{Ker}(\pi) = \{0\}$, and $\pi(1)$ is an element of infinite order in S^1. That is, $\pi(1) = exp(x2\pi i)$ where x is irrational in $[0,1)$; if x were rational, $\pi(1)$ would have finite order.

4. (a) $\pi(wz) = w^n z^n = \pi(w)\pi(z)$ for any w, z in S^1. $\text{Ker}(\pi) = \{exp(k2\pi i/n) | 0 \leq k < n\}$.

If π is one-one, then $\text{Ker}(\pi) = \{1\}$, and $n = \pm 1$. These choices of n also guarantee onto.

(b) $\pi(xy) = sgn(xy)|xy|^s = sgn(x)sgn(y)|x|^s|y|^s = \pi(x)\pi(y)$.

If $s = 0, n = 0$, $\text{Ker}(\pi) = \mathbb{R}^*$. If $s = 0, n = 1$, $\text{Ker}(\pi) = \{\text{positive reals }\}$. If $n = 0, s \neq 0$, $\text{Ker}(\pi) = \{\pm 1\}$. If $n = 1, s \neq 0$, then π is an isomorphism.

(c) $\pi(wz) = |wz|^s = |w|^s|z|^s = \pi(w)\pi(z)$.

If $s = 0$, then $\text{Ker}(\pi) = \mathbb{C}^*$. Otherwise $\text{Ker}(\pi) = S^1$; that is, all z such that $|z|^s = 1$, or $|z| = 1$.

(d) We note that $det : GL(2,\mathbb{R}) \to \mathbb{R}^*$ is a homomorphism, so we can compose with any homomorphism from (b) to get another homomorphism. Kernels are $GL(2,\mathbb{R})$, $SL(2,\mathbb{R})$, and $\{det(A) = \pm 1\}$, respectively.

5. Problem 3: n/d evenly spaced points on the circle, based at 1. Irrational x: the image is a countable, dense subset of S^1; that is, every point of the circle can be approximated by points in the image.

Problem 4(a): S^1 unless $n = 0$; 4(b): 1, $\{\pm 1\}$, {postivie reals}, $\mathbb{R}^*$; 4(c): {positive reals}; 4(d): 1, $\mathbb{R}^*$, {positive reals}.

6. (1) Note that $\pi : G \to \text{Im } \pi$ is a surjective homomorphism. Then the First Isomorphism Theorem follows from the theorem given in the video.

(2) Consider the natural map $\pi : H/(H \cap N) \to HN/N$ defined by $\pi(xH \cap N) = xN$. Note that $xH \cap N \subseteq xN \subseteq HN$.

We show the homomorphism, one-one, and onto properties. Suppose x, y are in H.

(1) Homomorphism: $\pi(xH \cap N)\pi(yH \cap N) = xNyN = xyN = \pi(xyH \cap N)$.
(2) One-one: suppose $\pi(xH \cap N) = \pi(yH \cap N)$. Then $xN = yN$, and $x^{-1}y$ is in $N \cap H$. So $xH \cap N = yH \cap N$, and
(3) Onto: for any xH in HN/N, $\pi(xH \cap N) = xN$.

(3) Normal: $(gK)(hK)(g^{-1}K) = ghg^{-1}K = h'K$ for some h' in K.

Isomorphism: First note that the coset of $gK(H/K)$ in $(G/K)/(H/K)$ is the set of K-cosets $\{ghK\}$ where h ranges over elements of H. Consider the isomorphism $\pi : (G/K)/(H/K) \to G/H$ defined by $\pi(gK(H/K)) = gH = \cup ghK$.

We show the homomorphism, one-one, and onto properties. Suppose x, y are in H.

(1) Homomorphism: $\pi(xK(H/K))\pi(yK(H/K)) = xHyH = xyH = \pi(xyK(H/K))$.
(2) One-one: suppose $\pi(xK(H/K)) = \pi(yK(H/K))$. Then $xH = yH$, and $x^{-1}y$ is in H. So $x^{-1}yK(H/K) = \{H/K\}$, and $xK(H/K) = yK(H/K)$, and
(3) Onto: for any xH in G/H, $\pi(xK(H/K)) = xH$.

7. Since $\text{Im}(\pi) \cong G/\text{Ker}(\pi)$ is abelian, the commutator subgroup of G is contained in $\text{Ker}(\pi)$. (3) and (4) are difficult without the commutator.

(1) $G = A_4$: the commutator subgroup is $H = \{e, (12)(34), (13)(24), (14)(23)\}$ and the abelianization is $G/H \cong \mathbb{Z}/3$. Only 2 homomorphisms.
(2) $G = S_4$: the commutator subgroup is A_4, and the abelianization is $\mathbb{Z}/2$. Only 2 homomorphisms
(3) $G = D_{2n}$: if n odd, the commutator subgroup is the rotation subgroup, and the abelianization is $\mathbb{Z}/2$. Only 2 homomorphisms. If n is even, the abelianization is $\mathbb{Z}/2 \times \mathbb{Z}/2$. Only 5 homomorphisms, one for each subgroup of the quotient, and
(4) $G = SL(2, \mathbb{R})$: the commutator is G itself, so only the trivial homomorphism into G/G, the one element group.

8. (a) Suppose $G = \langle g \rangle$. We show that $H = \langle \pi(g) \rangle$. If h is in H, then $h = \pi(x)$ for some x in G. But $x = g^k$, so $h = \pi(x) = \pi(g^k) = [\pi(g)]^k$.

(b) $\mathbb{Z}/4 \cong (\mathbb{Z}/5)^* = \{1, 2, 3, 4\}$ with generators 2 and 3. We can use $0, 1, 2, 3 \to 1, 2, 4, 3$ or $0, 1, 2, 3 \to 1, 3, 4, 2$.

$\mathbb{Z}/6 \cong (\mathbb{Z}/7)^* = \{1, 2, 3, 4, 5, 6\}$ with generators 3 and 5. We can use $0, 1, 2, 3, 4, 5 \to 1, 3, 2, 6, 4, 5$ or $0, 1, 2, 3, 4, 5 \to 1, 5, 4, 6, 2, 3$.

9. (a) We can inscribe an icosahedron inside of a dodecahedron by placing the 12 vertices at the center of each face.

(b) We can inscribe an octahedron inside a cube by placing the 6 vertices at the center of each face.

10. If a group is defined through generators and relations, an isomorphism sends generators to generators and preserves relations. Construct an isomorphism $\pi : G \to D_{2n}$ by defining

$$\pi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = r, \qquad \pi \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = c.$$

Then verify that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = I, \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^2 = I, \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Now π extends to elements of the form r^k and cr^k using the homomorphism property. Consistency of multiplication follows from the last relation.