Clebsch-Gordan Coefficients in Characteristic Zero

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m=3, n=4, k=0

$$(m+1) \text{ x } (m+n-2k+1) \;=\; 4 \text{ x } 8$$

$k = 0$: Pascal's Triangle (truncate at right corner for symmetry)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 0 & 0 \\ 0 & 0 & 1 & 3 & 6 & 10 & 15 & 0 \\ 0 & 0 & 0 & 1 & 4 & 10 & 20 & 35 \end{bmatrix}$$

m=3, n=4, k=1:

$$(m+1) \times (m+n-2k+1) \;=\; 4 \times 6$$

$k = 1$: Set left-most column - $k + 1 = 2$ nonzero entries,
Pascal's Recurrence to Right

$$\begin{bmatrix} 3 & 3 & 3 & 3 & 0 & 0 \\ -4 & -1 & 2 & 5 & 8 & 0 \\ 0 & -4 & -5 & -3 & 2 & 10 \\ 0 & 0 & -4 & -9 & -12 & -10 \end{bmatrix}$$

m=3, n=4, k=2

$(m+1)$ x $(m+n-2k+1)$ = 4 x 4

$k = 2$: Set left-most column - $k+1 = 3$ nonzero entries, Pascal's Recurrence to Right

$$\begin{bmatrix} 3 & 3 & 3 & 0 \\ -6 & -3 & 0 & 3 \\ 6 & 0 & -3 & -3 \\ 0 & 6 & 6 & 3 \end{bmatrix}$$

m=3, n=4, k=3

$(m+1)$ x $(m+n-2k+1)$ = 4 x 2

$k=3$: Set left-most column - $k+1=4$ nonzero entries, Pascal's Recurrence to Right

$$\begin{bmatrix} 1 & 1 \\ -2 & -1 \\ 3 & 1 \\ -4 & -1 \end{bmatrix}$$

First $k+1$ Entries Of The Left-Most Column

$$(-1)^s \binom{m-s}{k-s}\binom{n-k+s}{s}, \qquad\qquad 0 \le s \le k$$

$$k=1: \quad \left(\binom{3}{1}\binom{3}{0}, -\binom{2}{0}\binom{4}{1}\right) \to (3,-4,0,0)$$

$$k=2: \quad \left(\binom{3}{2}\binom{2}{0}, -\binom{2}{1}\binom{3}{1}, \binom{1}{0}\binom{4}{2}\right) \to (3,-6,6,0)$$

$$k=3: \quad \left(\binom{3}{3}\binom{1}{0}, -\binom{2}{2}\binom{2}{1}, \binom{1}{1}\binom{3}{2}, -\binom{0}{0}\binom{4}{3}\right)$$

$$\to (1,-2,3,-4)$$

The Algorithm

Given non-negative $m, \ n, \ k$ with $0 \leq k \leq min(m, n)$

1. Initialize matrix of size $(m + 1)$ by $(m + n - 2k + 1)$,
2. Initialize left-most column with binomial coefficient pattern,
3. Pascal's Recurrence to the Right,
4. Zero out upper-right column for symmetry.

Columns are coordinate vectors associated to a tensor product of vector spaces

Main Result

Fix m, n, k as before. The $(i+1, i+j-k+1)$ entry of the corresponding matrix in the algorithm equals

$$\begin{aligned} c_{m,n,k}(i,j) &= \sum_{s=0}^{k} (-1)^s \binom{i+j-k}{i-s} \binom{m-s}{k-s} \binom{n-k+s}{s} \\ &= \sum_{s=0}^{k} (-1)^s \binom{i+j-k}{i-s} \binom{m-i}{k-s} \binom{n-j}{s} \end{aligned}$$

Pascal's Recurrence:

$$c_{m,n,k}(i,j) = c_{m,n,k}(i-1,j) + c_{m,n,k}(i,j-1)$$

Generating Function

$$c_{m,n,k}(i,j) = \sum_{s=0}^{k} (-1)^s \binom{i+j-k}{i-s} \binom{m-s}{k-s} \binom{n-k+s}{s}$$

is the coefficient of $x^j y^i$ in the expansion of

$$(x+y)^{i+j-k} \cdot \frac{1}{(1-x)^{m-k+1}} \cdot \frac{1}{(1+y)^{n-k+1}}$$

This yields a second Pascal-type recurrence

$$c_{m,n,k}(i,j) = c_{m-1,n,k}(i-1,j) + c_{m,n-1,k}(i,j-1)$$

Clebsch-Gordan Coefficients

$$C_{m,n,k}(i,j) = \frac{\binom{m+n-k}{m-i,\ n-j,\ *}\ c_{m,n,k}(i,j)}{\binom{m+n-k}{i,\ j,\ *}\ D(m,n,k)} \in \mathbb{Q}$$

where

$$D(m,\ n,\ k) = \binom{m+n-k+1}{k}\ \binom{m+n-2k}{m-k}$$

Clebsch-Gordan Coefficients

1. Classical Invariant Theory
2. Tensor products of finite-dimensional representations of $SL(2, R), SU(2), SL(2, C)$
3. Quantum Theory of Angular Momentum
4. Multiplication Formula for Jacobi Polynomials and other Special Functions
5. Modular forms: Rankin-Cohen brackets

Clebsch-Gordan Decomposition

$V(m)$ = irreducible finite-dimensional representationf for $SU(2)$
Highest weight = m, dimension = $m + 1$

$0 \leq m \leq n$:

$$V(m) \otimes V(n) = \bigoplus_{k=0}^{m} V(m+n-2k)$$
$$= V(n-m) \oplus V(n-m+2) \oplus \cdots \oplus V(n+m)$$

Example: $k \;=\; 3,\; 2,\; 1,\; 0$

$$V(3) \otimes V(4) = V(1) \oplus V(3) \oplus V(5) \oplus V(7)$$

$$4 \cdot 5 = 2 + 4 + 6 + 8$$

Clebsch-Gordan Decomposition - Vector Level

$V(m)$: basis $\{u_i\}$

$V(n)$: basis $\{v_i\}$

Natural basis for $V(m) \otimes V(n)$: $\{u_i \otimes v_j\}$

Bases for each $V(m+n-2k)$: $\{w_p^k\}$

$$w_p^k = \sum_{i+j} c_{m,n,k}(i,j) u_i \otimes v_j$$

$$u_i \otimes v_j = \sum_k C_{m,n,k}(i,j) w_p^k$$

Wigner's Formula (1931)

$$C'_{m,n,k}(i,j) = \sqrt{\frac{(m+n-2k+1)(m+n-i-j-k)!(i+j-k)!i!j!k!}{(m-i)!(n-j)!(m-k)!(n-k)!(m+n-k+1)!}} \cdot S$$

where

$$S = \sum_{s=0}^{k} (-1)^{i-s} \frac{(m-i+s)!(n-k+i-s)!}{s!(i+j-k-s)!(i-s)!(k-i+s)!}$$

Traditionally studied through representation theory, Jacobi polynomials and hypergeometric series of type ${}_3F_2$

Wigner's Formula (1931)

With binomial coefficients and shift $s \to i - s$,

$$C'_{m,n,k}(i,j) \;=\; \sqrt{\frac{(m+n-2k+1)\binom{m+n-k}{m-i,\ n-j,\ *}}{(m+n-k+1)\ \binom{m+n-k}{m-k,\ n-k,\ k}\binom{m+n-k}{i,\ j,\ *}}}\cdot S$$

where

$$S = \sum_{s=0}^{k} (-1)^s \binom{i+j-k}{i-s}\binom{m-s}{k-s}\binom{n-k+s}{s}$$

Racah's Formula (1942)

$$C'_{m,n,k}(i,j) \;=\; \sqrt{\frac{(m+n-2k+1)\binom{m+n-k}{m-k,\;n-k,\;k}}{(m+n-k+1)\;\binom{m+n-k}{m-i,\;n-j,\;*}\binom{m+n-k}{i,\;j,\;*}}}\cdot S'$$

where

$$S' = \sum_{s=0}^{k} (-1)^s \binom{k}{s}\binom{m-k}{i-s}\binom{n-k}{j-k+s}.$$

Techniques:

Traditional:

1. ONB and normalizations
2. Special functions and hypergeometric series
3. weighted recurrences

Characteristic Zero:

1. no normalizations
2. Generating function and Binomial Identity
3. Pascal's recurrence

Combinatorial Toolbox:

1. highest weight vectors with integral coordinates (LMC)
2. Clebsch-Gordan coefficients with rational values
3. elementary generating function
4. recurrence relations ($144 = 2 \cdot 72$)
5. Regge symmetries (72 symmetries, subgroup of S_6)
6. Elementary algorithm for bulk computation

Open Problem:

When does $C_{m,n,k}(i,j) = 0$?