The gentle art of 3x3 semi-magic squares

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Full contact:

Stanley, R., Combinatorics and Commutative Algebra, 1996, 2nd Ed.

Louck, J. D., Applications of Unitary Symmetry and Combinatorics, 2011.

Gentle approach:

First principles, computer experimentation

Example: $J = 7$: Noiseless

$$\begin{array}{c}
\circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \\
\circ \quad \mathbf{1} \quad \cdot \quad \mathbf{3} \quad \cdot \quad \mathbf{1} \quad \circ \\
\circ \quad \cdot \quad \mathbf{2} \quad \mathbf{2} \quad \cdot \quad \circ \\
\circ \quad \mathbf{3} \quad \mathbf{2} \quad \mathbf{3} \quad \circ \\
\circ \quad \cdot \quad \cdot \quad \circ \\
\circ \quad \mathbf{1} \quad \circ \\
\circ \quad \circ \\
\circ
\end{array}$$

Example: $J = 13$: Noiseless

$\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$

$\circ$ **1** · **1** · **1** **6** **1** · **1** · **1** $\circ$

$\circ$ · **2** · · **2** **2** · · **2** · $\circ$

$\circ$ **1** · **3** **2** **5** **2** **3** · **1** $\circ$

$\circ$ · · **2** **4** **4** **2** · · $\circ$

$\circ$ **1** **2** **5** **4** **5** **2** **1** $\circ$

$\circ$ **6** **2** **2** **2** **2** **6** $\circ$

$\circ$ **1** · **3** · **1** $\circ$

$\circ$ · · · · $\circ$

$\circ$ **1** **2** **1** $\circ$

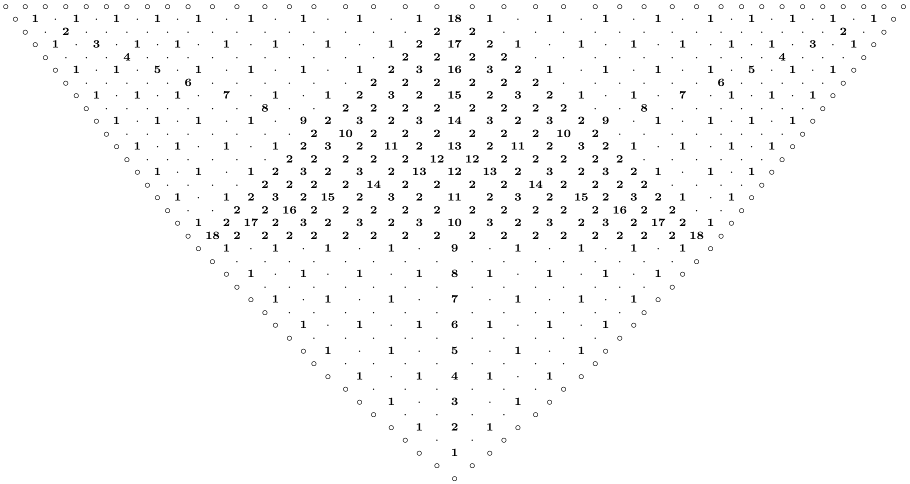
$\circ$ · · $\circ$

$\circ$ **1** $\circ$

$\circ$ $\circ$

$\circ$

Example: $J = 37$: Noiseless



Example: $J = 15$

○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○
○ **1** · **3** · **1** · **7** · **1** · **3** · **1** ○
○ · **2** · · · **2** **2** · · · **2** · ○
○ **3** · **5** · **3** **6** **3** · **5** · **3** ○
○ · · · **4** **2** **2** **4** · · · ○
○ **1** · **3** **2** **13** **2** **3** · **1** ○
○ · **2** **6** **2** **2** **6** **2** · ○
○ **7** **2** **3** **4** **3** **2** **7** ○
○ · · · · · · ○
○ **1** · **5** · **1** ○
○ · · · · ○
○ **3** **2** **3** ○
○ · · ○
○ **1** ○
○ ○
○

Example: $J = 19$

$$\begin{array}{c}
\circ \;\; \circ \;\; \circ \;\; \circ \;\; \circ \;\; \circ \;\; \circ \;\; \circ \;\; \circ \;\; \circ \;\; \circ \;\; \circ \;\; \circ \;\; \circ \;\; \circ \;\; \circ \;\; \circ \;\; \circ \;\; \circ \;\; \circ \\
\circ \;\; \mathbf{1} \;\; \cdot \;\; \mathbf{3} \;\; \mathbf{4} \;\; \mathbf{1} \;\; \cdot \;\; \mathbf{3} \;\; \cdot \;\; \mathbf{9} \;\; \cdot \;\; \mathbf{3} \;\; \cdot \;\; \mathbf{1} \;\; \mathbf{4} \;\; \mathbf{3} \;\; \cdot \;\; \mathbf{1} \;\; \circ \\
\circ \;\; \cdot \;\; \mathbf{2} \;\; \mathbf{2} \;\; \cdot \;\; \cdot \;\; \mathbf{2} \;\; \cdot \;\; \mathbf{2} \;\; \mathbf{2} \;\; \cdot \;\; \mathbf{2} \;\; \cdot \;\; \cdot \;\; \mathbf{2} \;\; \mathbf{2} \;\; \cdot \;\; \circ \\
\circ \;\; \mathbf{3} \;\; \mathbf{2} \;\; \mathbf{3} \;\; \mathbf{2} \;\; \mathbf{1} \;\; \cdot \;\; \mathbf{5} \;\; \mathbf{8} \;\; \mathbf{5} \;\; \cdot \;\; \mathbf{1} \;\; \mathbf{2} \;\; \mathbf{3} \;\; \mathbf{2} \;\; \mathbf{3} \;\; \circ \\
\circ \;\; \mathbf{4} \;\; \cdot \;\; \mathbf{2} \;\; \mathbf{4} \;\; \cdot \;\; \mathbf{4} \;\; \mathbf{2} \;\; \mathbf{2} \;\; \mathbf{4} \;\; \cdot \;\; \mathbf{4} \;\; \mathbf{2} \;\; \cdot \;\; \mathbf{4} \;\; \circ \\
\circ \;\; \mathbf{1} \;\; \cdot \;\; \mathbf{1} \;\; \cdot \;\; \mathbf{5} \;\; \mathbf{2} \;\; \mathbf{7} \;\; \mathbf{2} \;\; \mathbf{5} \;\; \cdot \;\; \mathbf{1} \;\; \cdot \;\; \mathbf{1} \;\; \circ \\
\circ \;\; \cdot \;\; \mathbf{2} \;\; \cdot \;\; \mathbf{4} \;\; \mathbf{2} \;\; \mathbf{6} \;\; \mathbf{6} \;\; \mathbf{2} \;\; \mathbf{4} \;\; \cdot \;\; \mathbf{2} \;\; \cdot \;\; \circ \\
\circ \;\; \mathbf{3} \;\; \cdot \;\; \mathbf{5} \;\; \mathbf{2} \;\; \mathbf{7} \;\; \mathbf{6} \;\; \mathbf{7} \;\; \mathbf{2} \;\; \mathbf{5} \;\; \cdot \;\; \mathbf{3} \;\; \circ \\
\circ \;\; \cdot \;\; \mathbf{2} \;\; \mathbf{8} \;\; \mathbf{2} \;\; \mathbf{2} \;\; \mathbf{2} \;\; \mathbf{2} \;\; \mathbf{8} \;\; \mathbf{2} \;\; \cdot \;\; \circ \\
\circ \;\; \mathbf{9} \;\; \mathbf{2} \;\; \mathbf{5} \;\; \mathbf{4} \;\; \mathbf{5} \;\; \mathbf{4} \;\; \mathbf{5} \;\; \mathbf{2} \;\; \mathbf{9} \;\; \circ \\
\circ \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \circ \\
\circ \;\; \mathbf{3} \;\; \mathbf{2} \;\; \mathbf{1} \;\; \mathbf{4} \;\; \mathbf{1} \;\; \mathbf{2} \;\; \mathbf{3} \;\; \circ \\
\circ \;\; \cdot \;\; \cdot \;\; \mathbf{2} \;\; \mathbf{2} \;\; \cdot \;\; \cdot \;\; \circ \\
\circ \;\; \mathbf{1} \;\; \cdot \;\; \mathbf{3} \;\; \cdot \;\; \mathbf{1} \;\; \circ \\
\circ \;\; \mathbf{4} \;\; \mathbf{2} \;\; \mathbf{2} \;\; \mathbf{4} \;\; \circ \\
\circ \;\; \mathbf{3} \;\; \mathbf{2} \;\; \mathbf{3} \;\; \circ \\
\circ \;\; \cdot \;\; \cdot \;\; \circ \\
\circ \;\; \mathbf{1} \;\; \circ \\
\circ \;\; \circ \\
\circ
\end{array}$$

$\mathbb{M}_3 =$ 3×3 weakly semi-magic squares

$$M = \begin{bmatrix} a & b & k \\ r & * & * \\ * & * & c \end{bmatrix}$$

1. all entries are nonnegative integers,
2. all line sums along rows and columns are equal, and
3. this sum, the **magic number**, equals $J = a + b + k$.

Also called **integer "doubly-stochastic" matrices**.
Particle physics: **Regge symbol**

$$\mathbb{M}_3(J) = 3 \times 3 \text{ magic squares with magic number } J$$

$$\mathbb{M}_3 = \cup_{J \geq 0} \mathbb{M}_3(J)$$

Determinantal Symmetries

For 3x3 matrices, let G be the group of determinantal symmetries;

that is,

1. G is generated by row switches, column switches, and transpose,
2. every element g of G may be expressed uniquely as

$$g = R(\sigma)C(\tau)T^{\epsilon} \quad \text{with} \quad \sigma, \tau \in S_3,$$

   and
3. $|G| = 72.$

These symmetries preserve

1. the semi-magic square property,
2. the magic number J, and thus
3. each $\mathbb{M}_3(J)$.

Partition of $\mathbb{M}_3$: Triangle and Hexagon

First step:
Partition $\mathbb{M}_3 = \cup_{J \geq 0} \mathbb{M}_3(J)$.

Second Step:
Partition $\mathbb{M}_3(J)$ into subsets of magic squares with a fixed top line.

A fixed top line corresponds to an ordered partition of J into 3 parts (possibly with zeros).

We use k, the third top line entry to denote level.

Third Step: All magic squares for a fixed top line are parametrized by a (possibly degenerate) hexagon.

Square: rows/columns 0 through J,
position indicates remaining variables (r, c)

Example: Triangle

$$\mathbf{J = 4}$$

$$(0,4,0),(1,3,0),(2,2,0),(3,1,0),(4,0,0)$$

$$(0,3,1),(1,2,1),(2,1,1),(3,0,1)$$

$$(0,2,2),(1,1,2),(2,0,2)$$

$$(0,1,3),(1,0,3)$$

$$(0,0,4)$$

Column switches induce S_3 symmetries of the triangle

Hexagon

Fix $\quad a,\ b,\ k.$

$$M = \begin{bmatrix} a & b & k \\ r & * & * \\ * & * & c \end{bmatrix}$$

$$0 \le r \le b + k, \qquad 0 \le c \le a + b,$$

$$-a \le r - c \le k.$$

Example: Hexagon

$\mathbf{J} = \mathbf{4}$, top line: (0, 4, 0) $\rightarrow$ 5 squares

$$\begin{bmatrix} 0 & 4 & 0 \\ r & 0 & * \\ * & 0 & c \end{bmatrix} \mapsto \begin{bmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & \boxed{*} & \\ & & & & * \end{bmatrix}$$

Box corresponds to $(r, c) = (3, 3)$:

$$\begin{bmatrix} 0 & 4 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Example: Hexagon

$\mathbf{J} = \mathbf{4}$, top line: (3, 0, 1) $\rightarrow$ 8 squares

$$\begin{bmatrix} 3 & 0 & 1 \\ r & * & * \\ * & * & c \end{bmatrix} \mapsto \begin{bmatrix} * & * & * & * \\ * & * & \boxed{*} & * \\ & & & \\ & & & \end{bmatrix}$$

Box corresponds to $(r, c) = (1, 2)$:

$$\begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

Example: Hexagon

$$\mathbf{J} = \mathbf{4}, \qquad \text{top line: } (1,\ 2,\ 1) \quad \rightarrow \quad 10 \text{ squares}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ r & * & * \\ * & * & c \end{bmatrix} \mapsto \begin{bmatrix} * & * & & & \\ * & * & * & & \\ & * & \boxed{*} & * & \\ & & * & * & \\ & & & & \end{bmatrix}$$

Box corresponds to $(r, c) = (2, 2)$:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Hexagon for Top Line (a, b, k):

J, top line: (a, b, k)

Polygon is a hexagon at $(0, 0)$ with

1. vertical side: $k + 1$
2. diagonal side: $b + 1$
3. horizontal side: $a + 1$

$$\begin{aligned} |(a, b, k)| &= (a + b + 1)(b + k + 1) - b(b + 1) \\ &= 1 + (a + b + k) + (ab + ak + bk) \end{aligned}$$

Column switches re-orient the polygon

Enumeration

McMahon's formula for $\mathbb{M}_3(J)$ (1915):

$$H_3(J) = \binom{4+J}{4} + \binom{3+J}{4} + \binom{2+J}{4}$$

$$= \frac{(J+1)(J+2)(J^2+3J+4)}{8}$$

Number of top lines for J:

$$1 + 2 + 3 + \cdots + (J+1) = \frac{(J+2)(J+1)}{2}$$

Number of magic squares with top line (a, b, k):

$$1 + (a + b + k) + (ab + ak + bk)$$

Example: Triangle

$$\mathbf{J = 4}$$

$$5,\ \ 8,\ \ 9,\ \ 8,\ \ 5$$

$$8,\ 10,\ 10,\ 8$$

$$9,\ 10,\ 9$$

$$8,\ 8$$

$$5$$

$$Sum = 120 = \frac{(4+1)(4+2)(4^2+3(4)+4)}{8}$$

Clebsch-Gordan coefficients/ function

$$C : \mathbb{M}_3 \to \mathbb{Z}$$

$$M = \begin{bmatrix} a & b & k \\ r & m & * \\ * & * & c \end{bmatrix} \mapsto$$

$$C(M) = \sum_{l=0}^{k} (-1)^l \binom{c}{r-l} \binom{b+k-l}{b} \binom{a+l}{a}$$

$C(M)$ is the coefficient of

$$x^m y^r$$

in the power series expansion of

$$\frac{(x+y)^c}{(1-x)^{b+1}(1+y)^{a+1}}$$

Example of Values for $J = 4$:

$$(0,4,0) \qquad (3,0,1) \qquad (1,2,1)$$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 & \\ -4 & -3 & -2 & -1 & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}, \begin{bmatrix} 3 & 3 & & & \\ -2 & 1 & 4 & & \\ & -2 & -1 & 3 & \\ & & -2 & -3 & \\ & & & & \end{bmatrix}$$

Finite differences $\rightarrow$ Binomial transforms (Pascal's Recurrence),
Riordan arrays.

Classification of Zeros:

Open problem: Classify the zeros of $C(M)$
(Biedenharn, Brudno, Louck, K. S. Rao)

Visual data?

We can use the partition as model for vanishing portraits of $C(M)$.

In the triangle for J, replace magic square counts with zero counts.

Example: $J = 8$

$$\begin{array}{ccccccccc}
9 & 16 & 21 & 24 & 25 & 24 & 21 & 16 & 9 \\
16 & 22 & \boxed{26} & 28 & 28 & 26 & 22 & 16 & \\
21 & 26 & 29 & 30 & 29 & 26 & 21 & & \\
24 & 28 & 30 & 30 & 28 & 24 & & & \\
25 & 28 & 29 & 28 & 25 & & & & \\
24 & 26 & 26 & 24 & & & & & \\
21 & 22 & 21 & & & & & & \\
16 & 16 & & & & & & & \\
9 & & & & & & & &
\end{array}$$

$$\begin{array}{ccccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \cdot & \boxed{2} & \cdot & \cdot & 2 & \cdot & \circ & \\
\circ & 2 & 2 & \cdot & 2 & 2 & \circ & & \\
\circ & \cdot & \cdot & \cdot & \cdot & \circ & & & \\
\circ & \cdot & 2 & \cdot & \circ & & & & \\
\circ & 2 & 2 & \circ & & & & & \\
\circ & \cdot & \circ & & & & & & \\
\circ & \circ & & & & & & & \\
\circ & & & & & & & &
\end{array}$$

$$H_3(8) = \frac{(10)(9)(64 + 24 + 4)}{8} = 1035, \qquad |Zeros| = 18,$$

Example: $J = 8$

Top line $(2,5,1)$: 26 squares, 2 zeros

$$\begin{bmatrix} 6 & 6 & 6 & & & & & \\ -3 & 3 & 9 & 15 & & & & \\ & -3 & \boxed{0} & 9 & 24 & & & \\ & & -3 & -3 & 6 & 30 & & \\ & & & -3 & -6 & \boxed{0} & 30 & \\ & & & & -3 & -9 & -9 & 21 \\ & & & & & -3 & -12 & -21 \end{bmatrix}$$

Zeros: $(r,c) \;=\; (2,2),\; (4,5)$

$$\begin{bmatrix} 2 & 5 & 1 \\ 2 & 1 & 5 \\ 4 & 2 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 5 & 1 \\ 4 & 2 & 2 \\ 2 & 1 & 5 \end{bmatrix}$$

Zero Counts

J	Zeros	J	Zeros	J	Zeros	J	Zeros	J	Zeros
1	0	**13**	99	**25**	558	**37**	945	**49**	2484
2	0	**14**	144	**26**	288	**38**	792	**50**	1044
3	1	**15**	154	**27**	793	**39**	2098	**51**	2665
4	0	**16**	0	**28**	0	**40**	0	**52**	0
5	9	**17**	252	**29**	927	**41**	1746	**53**	2673
6	0	**18**	0	**30**	0	**42**	0	**54**	1260
7	18	**19**	333	**31**	792	**43**	1611	**55**	3834
8	18	**20**	324	**32**	**432**	**44**	1332	**56**	1422
9	46	**21**	433	**33**	856	**45**	2035	**57**	2746
10	0	**22**	0	**34**	828	**46**	0	**58**	0
11	99	**23**	558	**35**	1719	**47**	2340	**59**	4131
12	0	**24**	**252**	**36**	0	**48**	**504**	**60**	0

Example: $J = 14$, $\qquad J = 20$

```
○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○
 ○ · 2 · 4 2 · · 2 4 · 2 · ○
  ○ 2 · 2 6 · 4 · 6 2 · 2 ○
   ○ · 2 · · 2 2 · · 2 · ○
    ○ 4 6 · 8 · 8 · 6 4 ○
     ○ 2 · 2 · · 2 · 2 ○
      ○ · 4 2 8 2 4 · ○
       ○ · · · · · · ○
        ○ 2 6 · 6 2 ○
         ○ 4 2 2 4 ○
          ○ · · · ○
           ○ 2 2 ○
            ○ · ○
             ○ ○
              ○
```

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○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○
 ○ · 2 · · 2 6 · 2 · · 2 · 6 2 · · 2 · ○
  ○ 2 · · 4 6 2 2 · · · 2 2 6 4 · · 2 ○
   ○ · · 4 6 2 2 2 · · 2 2 2 6 4 · · ○
    ○ · 4 6 4 2 2 · 4 · 2 2 4 6 4 · ○
     ○ 2 6 2 2 8 · · · · 8 2 2 6 2 ○
      ○ 6 2 2 2 · 4 · 4 · 2 2 2 6 ○
       ○ · 2 2 · · · · · · 2 2 · ○
        ○ 2 · · 4 · 4 · 4 · · 2 ○
         ○ · · · · · · · · · · ○
          ○ · · 2 2 8 2 2 · · ○
           ○ 2 2 2 2 2 2 2 2 ○
            ○ · 2 2 4 2 2 · ○
             ○ 6 6 6 6 6 6 ○
              ○ 2 4 4 4 2 ○
               ○ · · · · ○
                ○ · · · ○
                 ○ 2 2 ○
                  ○ · ○
                   ○ ○
                    ○
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Example: $J = 24$

```
. . . . . . . . . . . . . . . . . . . . . . . . .
 . . . . 4 . . . . 4 . . . . 4 . . . . 4 . . . .
  . . . 6 . . . . 6 2 . . . 2 6 . . . . 6 . . .
   . . 6 2 . . . 4 2 2 . . 2 2 4 . . . 2 6 . .
    . 4 . . 2 . 2 . 2 . . . 2 . 2 . 2 . . 4 .
     . . . . . . . . . . . . . . . . . . . .
      . . . . 2 . . . . 4 . . . . 2 . . . .
       . . . 4 . . . . 2 2 . . . . 4 . . .
        . . 6 2 2 . . 2 12 2 . . 2 2 6 . .
         . 4 2 2 . . 4 2 2 4 . . 2 2 4 .
          . . . . . . . . . . . . . . .
           . . . . . . . . . . . . . .
            . . . 2 2 . . . 2 2 . . .
             . . 2 2 . . . . 2 2 . .
              . 4 6 4 2 . 2 4 6 4 .
               . . . . . . . . . .
                . . . . 2 . . . .
                 . . . . . . . .
                  . . . 2 . . .
                   . 4 6 6 4 .
                    . . . . .
                     . . . .
                      . . .
                       . .
                        .
```

Example: $J = 24$: Top Line $(8, 8, 8)$, 12 zeros

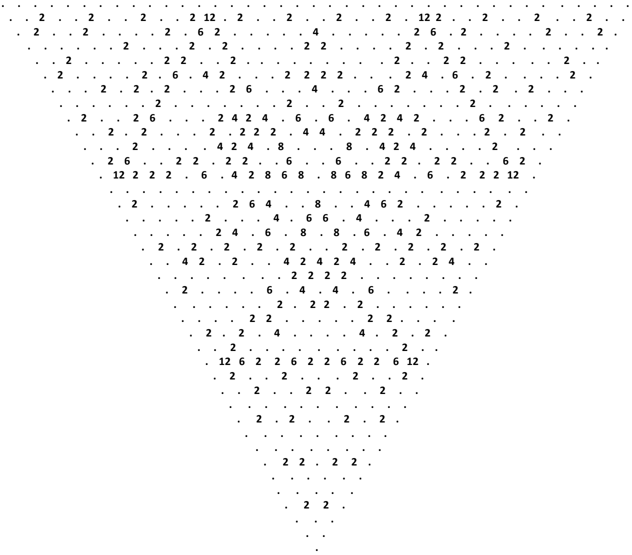
12870	12870	12870	12870	12870	12870	12870	12870	12870								
-57915	-45045	-32175	-19305	-6435	6435	19305	32175	45045	57915							
135135	77220	32175	0	-19305	-25740	-19305	0	32175	77220	135135						
-212355	-77220	0	32175	32175	12870	-12870	-32175	-32175	0	77220	212355					
245025	32670	-44550	-44550	-12375	19800	32670	19800	-12375	-44550	-44550	32670	245025				
-212355	32670	65340	20790	-23760	-36135	-16335	16335	36135	23760	-20790	-65340	-32670	212355			
135135	-77220	-44550	20790	41580	17820	-18315	-34650	-18315	17820	41580	20790	-44550	-77220	135135		
-57915	77220	0	-44550	-23760	17820	35640	17325	-17325	-35640	-17820	23760	44550	0	-77220	57915	
12870	-45045	32175	32175	-12375	-36135	-18315	17325	34650	17325	-18315	-36135	-12375	32175	32175	-45045	12870
	12870	-32175	0	32175	19800	-16335	-34650	-17325	17325	34650	16335	-19800	-32175	0	32175	-12870
		12870	-19305	-19305	12870	32670	16335	-18315	-35640	-18315	16335	32670	12870	-19305	-19305	12870
			12870	-6435	-25740	-12870	19800	36135	17820	-17820	-36135	-19800	12870	25740	6435	-12870
				12870	6435	-19305	-32175	-12375	23760	41580	23760	-12375	-32175	-19305	6435	12870
					12870	19305	0	-32175	-44550	-20790	20790	44550	32175	0	-19305	-12870
						12870	32175	32175	0	-44550	-65340	-44550	0	32175	32175	12870
							12870	45045	77220	77220	32670	-32670	-77220	-77220	-45045	-12870
								12870	57915	135135	212355	245025	212355	135135	57915	12870

$$M = \begin{bmatrix} 8 & 8 & 8 \\ 2 & 9 & 13 \\ 14 & 7 & 3 \end{bmatrix}$$ has $det(M) = 0$ but no matching rows/columns.

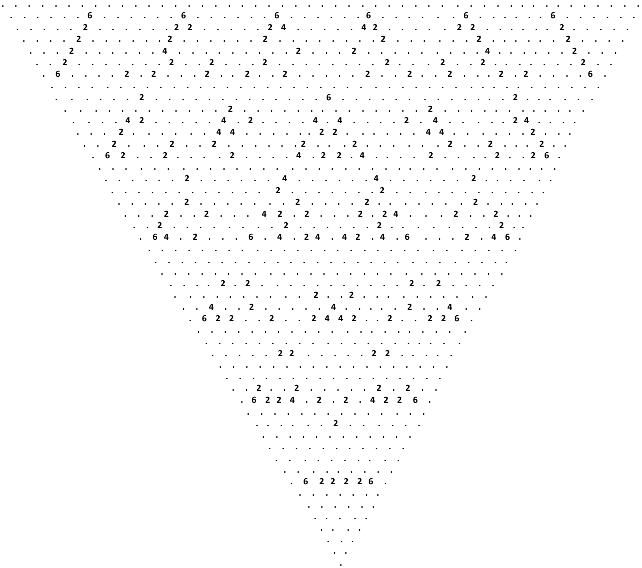
Example: $J = 32$

```
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
 . . 2 . . 2 . . 2 . 10 2 . . 2 . . 2 . . 2 10 . 2 . . 2 . . 2 . .
  . 2 . . 2 . . 2 . 2 6 2 . . . . . . . 2 6 2 . 2 . . 2 . . 2 .
   . . . . . . 2 . 2 . . . . . . . . . . . . 2 . 2 . . . . . .
    . . 2 . . . . 2 2 . 2 . . . . . . . 2 . 2 2 . . . . 2 . .
     . 2 . . . . 2 . . 2 2 . . . . . . 2 2 . . 2 . . . . 2 .
      . . . 2 . 2 4 2 . . 2 2 2 . 2 2 2 . . 2 4 2 . 2 . . .
       . . 2 . 2 . 2 . . . 4 2 . . 2 4 . . . 2 . 2 . 2 . .
        . 2 . 2 2 . . . 4 . 2 . . . 2 . 4 . . . 2 2 . 2 .
         . . 2 . . 2 . . . . . . . . . . . . 2 . . 2 . .
          . 10 6 . 2 2 2 4 2 . 4 . 4 . 2 4 2 2 2 . 6 10 .
           . 2 2 . . . 2 2 . . . . . . 2 2 . . . 2 2 .
            . . . . . . 2 . . . 4 . . . 2 . . . . . .
             . . . . . . . . . . . . . . . . . . . .
              . 2 . . . . 2 2 2 . 2 2 2 . . . . 2 .
               . . . . . . 2 4 . . 4 2 . . . . . .
                . . . . . . 2 . 4 . 2 . . . . . .
                 . 2 . . . 2 . . . . 2 . . . 2 .
                  . . . . 2 2 . . . 2 2 . . . .
                   . . 2 . . . 2 2 . . . 2 . .
                    . 2 6 . 2 . 4 . 2 . 6 2 .
                     . 10 2 2 2 2 2 2 2 2 10 .
                      . . . . . . . . . . .
                       . 2 2 2 . . 2 2 2 .
                        . . . . . . . . .
                         . . . . . . . .
                          . 2 2 . 2 2 .
                           . . . . . .
                            . . . . .
                             . 2 2 .
                              . . .
                               . .
                                .
```

Example: $J = 38$



Example: $J = 48$



Thank you!