The generatingfunctionology of Clebsch-Gordan coefficients

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Preamble

Wilf, Herbert (1931-2012) , “generatingfunctionology” (book), 1990, 1994, 2006.

This talk: overview of three preprints on arXiv:

1. (with W. G. Kim), Proceedings of Gestur Olafsson’s 65th birthday conference,
2. Proceedings of CANT 2017/2018,
3. arXiv (May 2019), and
4. work in progress (magic squares).

Also see pdfs of recent talks at mathdoctorbob.org
under Group Theory Courses

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Motivating Question:

Fix $m, \ n \geq 0$.

$V(n)$ be the irr. rep. of $SL(2, \mathbb{C})$ with highest weight n.

Then $dim_{\mathbb{C}} V(n) = n + 1$.

Clebsch-Gordan Decomposition (Gap 2 - Multiplicity One)

$$V(m) \otimes V(n) \quad \cong \quad V(|m-n|) \oplus \cdots \oplus V(m+n-2) \oplus V(m+n)$$

with no omissions.

Suppose $m, \ n$ are even, and non-zero $\phi_m^0, \ \phi_n^0$ have weight zero. Then

Spherical Tensoring (Gap 4)

$$\phi_m^0 \otimes \phi_n^0 \quad \in \quad V(|m-n|) \oplus \cdots \oplus V(m+n-4) \oplus V(m+n)$$

with no omissions.

(Half Proof: Weyl group element changes parity of ϕ_m^0 by $(-1)^{m/2}$)

Clebsch-Gordan Coefficients for $SU(2)$

1. Classical Invariant Theory:
   A. Clebsch; P. Gordan (Erlangen)
2. Quantum Mechanics (Weyl, Wigner):
   Coupling of angular momentum and spin
   Squared CGCs as probabilities
3. Closed formulas: Wigner, Racah
4. Symmetries using Magic Squares: Regge
5. Vanishing theory: Biedenharn, Louck, Rao, Raynal et al
6. Simulations in nuclear physics/ chemistry

Example: $V(2) \otimes V(2) \to V(4)$

Let e, f, h be the usual basis for the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$
Let ϕ_2 be a highest weight vector for $V(2)$

Using the Leibniz rule, a basis for target $V(4)$ is given by

$$
\begin{aligned}
\phi_{2,2} &= \phi_2 \otimes \phi_2, \\
f^1(\phi_{2,2}) &= \phi_2 \otimes f\phi_2 \quad + \quad f\phi_2 \otimes \phi_2, \\
f^2(\phi_{2,2}) &= \phi_2 \otimes f^2\phi_2 \; + 2f\phi_2 \otimes f\phi_2 \quad + \quad f^2\phi_2 \otimes \phi_2, \\
f^3(\phi_{2,2}) &= \qquad\qquad 3f\phi_2 \otimes f^2\phi_2 + 3f^2\phi_2 \otimes f\phi_2, \\
f^4(\phi_{2,2}) &= \qquad\qquad\qquad\qquad 6f^2\phi_2 \otimes f^2\phi_2.
\end{aligned}
$$

For this talk: Clebsch-Gordan coefficients without normalization for ONB.

That is, we drop unitarity to highlight combinatorial features.

Example: $V(2) \otimes V(2) \to V(4)$

1. array of size $dim(first)$ by $dim(target) \to 3 \times 5$.
2. columns as coordinate vectors (descending weight),
3. Leibniz rule $\to$ Pascal's identity (capital-L to right)

$$\begin{bmatrix} 1 & 1 & 1 & & \\ & 1 & 2 & 3 & \\ & & 1 & 3 & 6 \end{bmatrix}$$

Example: $V(2) \otimes V(2) \to V(2)$

Choose highest weight vector $\phi'_{2,2} = \phi_2 \otimes f\phi_2 - f\phi_2 \otimes \phi_2$.

Using the Leibniz rule, a basis is given by

$$
\begin{aligned}
\phi'_{2,2} &= \phi_2 \otimes f\phi_2 - f\phi_2 \otimes \phi_2, \\
f^1(\phi'_{2,2}) &= \phi_2 \otimes f^2\phi_2 - f^2\phi_2 \otimes \phi_2, \\
f^2(\phi'_{2,2}) &= f\phi_2 \otimes f^2\phi_2 - f^2\phi_2 \otimes f\phi_2.
\end{aligned}
$$

Example: $V(2) \otimes V(2) \to V(2)$

1. array of size *dim*(*first*) by *dim*(*target*) $\to 3 \times 3$.
2. columns as coordinate vectors (descending weight),
3. Leibniz rule $\to$ Pascal's identity (capital-L to right)

$$\begin{bmatrix} 1 & 1 & \\ -1 & 0 & 1 \\ & -1 & -1 \end{bmatrix}$$

Cartan Component: $V(m) \otimes V(n) \to V(m+n)$

Let ϕ_m a highest weight vector for $V(m)$.

The same process produces an array of size $m+1$ by $m+n+1$.

First column represents highest weight vector:

$$\phi_{m,n} = \phi_m \otimes \phi_n.$$

Leibniz rule: repeatedly apply f to highest weight vector.

Clebsch-Gordan coefficents fill a parallelogram section of Pascal's triangle.

Example: $V(3) \otimes V(4) \to V(7)$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & & & \\ & 1 & 2 & 3 & 4 & 5 & & \\ & & 1 & 3 & 6 & 10 & \boxed{15} & \\ & & & 1 & 4 & 10 & \boxed{20} & 35 \end{bmatrix}$$

Decode 7-th column: weight vector of weight $7 - 2(6) = -5$

1. need 6 powers of f
2. row 3 means start with f^2.

$$f^6(\phi_3 \otimes \phi_4) = 15\ f^2\phi_3 \otimes f^4\phi_4 + 20\ f^3\phi_3 \otimes f^3\phi_4.$$

Sub-Cartan Component: $V(m) \otimes V(m) \to V(2m-2)$

Let ϕ_m a highest weight vector for $V(m)$.

The same process produces an array of size $m+1$ by $2m-1$.

First column represents highest weight vector

$$\phi'_{m,m} = \phi_m \otimes f\phi_m - f\phi_m \otimes \phi_m.$$

Leibniz rule: repeatedly apply f to highest weight vector

Clebsch-Gordan coefficents fill a hexagon of the Catalan "triangle" (after Kirillov-Melnikov).

Catalan Number Triangle

$$\begin{bmatrix} \boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ -1 & 0 & \boxed{1} & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 0 & -1 & -1 & 0 & \boxed{2} & 5 & 9 & 14 & 20 & \cdots \\ 0 & 0 & -1 & -2 & -2 & 0 & \boxed{5} & 14 & 28 & \cdots \\ 0 & 0 & 0 & -1 & -3 & -5 & -5 & 0 & \boxed{14} & \cdots \\ 0 & 0 & 0 & 0 & -1 & -4 & -9 & -14 & -14 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Catalan Numbers (to the right of the diagonal zeros)

$1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ \ldots$

Catalan Number Triangle

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 0 & -1 & -1 & 0 & 2 & 5 & 9 & 14 & 20 & \cdots \\ 0 & 0 & -1 & -2 & -2 & 0 & 5 & 14 & 28 & \cdots \\ 0 & 0 & 0 & -1 & -3 & -5 & -5 & 0 & 14 & \cdots \\ 0 & 0 & 0 & 0 & -1 & -4 & -9 & -14 & -14 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$\textit{First column} \rightarrow p(x) = 1 - x$$

$$\begin{aligned} (1-x)(1+x) &= 1 + 0x - x^2 \\ (1-x)(1+x)^2 &= 1 + x - x^2 - x^3 \\ (1-x)(1+x)^3 &= 1 + 2x + 0x^2 - 2x^3 - x^4 \\ (1-x)(1+x)^4 &= 1 + 3x + 2x^2 - 2x^3 - 3x^4 - x^5 \end{aligned}$$

Example:

$V(3) \otimes V(3) \to V(4): \quad 4 \times 5$

$$3\begin{bmatrix} 1 & 1 & 1 & & \\ -1 & 0 & 1 & 2 & \\ & -1 & -1 & 0 & 2 \\ & & -1 & -2 & -2 \end{bmatrix}$$

$V(4) \otimes V(4) \to V(6): \quad 5 \times 7$

$$4\begin{bmatrix} 1 & 1 & 1 & 1 & & & \\ -1 & 0 & 1 & 2 & 3 & & \\ & -1 & -1 & 0 & 2 & 5 & \\ & & -1 & -2 & -2 & 0 & 5 \\ & & & -1 & -3 & -5 & -5 \end{bmatrix}$$

Hexagon: $V(m) \otimes V(n) \to V(m+n-2k)$

Given non-negative $m, \ n, \ k$ with $0 \leq k \leq min(m,n)$

1. Initialize matrix of size $dim(first)$ by $dim(target)$,
2. Initialize left-most column with highest weight condition,
3. Pascal's Recurrence (Capital-L),
4. Remove upper-right triangle for symmetry.

Columns are coordinate vectors for weight vectors of target space (descending weights)

Generating Function for Highest Weight Coefficients

The highest weight vector $\phi^k_{m,n}$ for $V(m+n-2k)$ is a linear combination of terms

$$\phi^k_{m,n} = \sum_{i+j=k} c(i,j)\ f^i\phi_m \otimes f^j\phi_n.$$

The coefficient of

$$f^i\phi_m \otimes f^j\phi_n \qquad \text{in} \qquad \phi^k_{m,n}$$

is the coefficient of $x^j y^i$ in the expansion of

$$\frac{1}{(1-x)^{m-k+1}} \cdot \frac{1}{(1+y)^{n-k+1}}.$$

Specifically, in row $i+1$

$$c_{m,n,k}(i,k-i) = (-1)^i \binom{m-i}{k-i} \binom{n-k+i}{i}$$

Generating Function for CG Coefficients

The vector $f^t\phi^k_{m,n}$ for $V(m+n-2k)$ is a linear combination of terms

$$f^t\phi^k_{m,n} = \sum_{i+j=t+k} c(i,j)\ f^i\phi_m \otimes f^j\phi_n.$$

The coefficient of

$$f^i\phi_m \otimes f^j\phi_n \qquad \text{in} \qquad f^t\phi^k_{m,n}$$

is the coefficient of $x^j y^i$ in the expansion of

$$(x+y)^t \cdot \frac{1}{(1-x)^{m-k+1}} \cdot \frac{1}{(1+y)^{n-k+1}}.$$

Specifically, in row $i+1$ and column $i+j-k+1$,

$$c_{m,n,k}(i,j) = \sum_{s=0}^{k} (-1)^s \binom{i+j-k}{i-s} \binom{m-s}{k-s} \binom{n-k+s}{s}$$

Wigner's Formula (1931)

For instance, Vilenkin's book, "Special Functions..." (1965).

$$C_{m,n,k}(i,j) = \sqrt{\frac{(m+n-2k+1)(m+n-i-j-k)!(i+j-k)!i!j!k!}{(m-i)!(n-j)!(m-k)!(n-k)!(m+n-k+1)!}} \cdot S$$

where

$$S = \sum_{s=0}^{k} (-1)^{i-s} \frac{(m-i+s)!(n-k+i-s)!}{s!(i+j-k-s)!(i-s)!(k-i+s)!}$$

Traditionally studied through representation theory, Jacobi polynomials and hypergeometric series of type ${}_3F_2$ (terminating)

Wigner's Formula (1931)

With binomial coefficients and shift $s \to i - s$,

$$C_{m,n,k}(i,j) \;=\; \sqrt{\frac{(m+n-2k+1)\binom{m+n-k}{m-i,\;n-j,\;*}}{(m+n-k+1)\;\binom{m+n-k}{m-k,\;n-k,\;k}\binom{m+n-k}{i,\;j,\;*}}}\cdot S$$

where

$$S = \sum_{s=0}^{k} (-1)^s \binom{i+j-k}{i-s}\binom{m-s}{k-s}\binom{n-k+s}{s}$$

Techniques:

Traditional:

1. ONB and normalizations
2. Special functions and hypergeometric series
3. Weighted recurrences

Characteristic zero, assuming highest weight rep's:

1. No normalizations $\rightarrow$ Integer coefficients
2. Uniform generating function
3. Pascal's recurrence (Calculus of finite differences)

See quantum CGC theory (Kirillov-Reshetikhin (1988) *et al)*

Combinatorial Toolbox:

Uniform generating function $\rightarrow$

1. Recurrence relations (three and four term)
2. Explicit calculations of summations using Dixon's formula
3. Elementary algorithm for programming (MAPLE and EXCEL)
4. **Orthogonality relations**
5. **Regge symmetries (magic squares)**

Topic: Orthogonality Relations

Change between orthonormal bases $\to$ unitary matrix

$$\sum_{j=1}^{n} c_{i,j}\overline{c_{i',j}} = \delta_{i,i'}$$

Non-degenerate invariant bilinear forms:
if X is in $\mathfrak{sl}(2,\mathbb{C})$, then

$$\langle Xu, v\rangle = -\langle u, Xv\rangle.$$

For example,

$$\langle f^i\phi_m, f^{m-i}\phi_m\rangle = (-1)^i\langle \phi_m, f^m\phi_m\rangle \neq 0$$

and

$$f^i\phi_m \otimes f^j\phi_n \qquad \text{pairs with} \qquad f^{m-i}\phi_m \otimes f^{n-j}\phi_n.$$

Orthogonality Relations (Non-vanishing Part)

The sum

$$\sum_{i+j=s} c_{m,n,k}(i,j)\ c_{m,n,k}(m-i,n-j)$$

is independent of s.

Example:

$V(3) \otimes V(3) \to V(4):$

$$\begin{bmatrix} \boxed{1} & 1 & 1 & * & \boxed{*} \\ \boxed{-1} & 0 & 1 & 2 & \boxed{*} \\ \boxed{0} & -1 & -1 & 0 & \boxed{2} \\ \boxed{0} & 0 & -1 & -2 & \boxed{-2} \end{bmatrix}, \quad \begin{bmatrix} 1 & \boxed{1} & 1 & \boxed{*} & * \\ -1 & \boxed{0} & 1 & \boxed{2} & * \\ 0 & \boxed{-1} & -1 & \boxed{0} & 2 \\ 0 & \boxed{0} & -1 & \boxed{-2} & -2 \end{bmatrix}$$

$$(1,-1,0,0)\cdot(-2,2,*,*) = -4$$
$$(1,0,-1,0)\cdot(-2,0,2,*) = -4$$
$$(1,1,-1,-1)\cdot(-1,-1,1,1) = -4$$

Generalized Binomial Transform

$$\{a_i\}_{i=0}^{\infty} \to p(x) = \sum_{i=0}^{\infty} a_i x^i$$

Generalized Binomial Transform

$$B^n a_k = a_{k,n} = \sum_{i=0}^{n} \binom{n}{i} a_{k-i}$$

$$\sum_{k=0}^{\infty} B^n a_k \; x^k = (1+x)^n \sum_{i=0}^{\infty} a_i x^i$$

Extend to all n using

$$(1+x)^{-n} = \sum_{i=0}^{\infty} (-1)^i \binom{n+i-1}{i} x^i$$

Pascal's Recurrence

$$n = 1 : B^1 a_k = a_{k-1} + a_k$$
$$n = 2 : B^2 a_k = a_{k-2} + 2a_{k-1} + a_k$$
$$n = 3 : B^3 a_k = a_{k-3} + 3a_{k-2} + 3a_{k-1} + a_k$$

Pascal's Recurrence (Capital L)

$$B^{n+1} a_k = B^n a_k + B^n a_{k-1}$$

Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Matrix Implementation (Binomial Array)

1. Fourth quadrant matrix
2. $\{a_i\}$ down first column, a_0 along first row
3. Pascal's recurrence: Capital-L summation

$$\begin{bmatrix} a_0 & a_0 & a_0 & a_0 & a_0 & \cdots \\ a_1 & a_0 + a_1 & 2a_0 + a_1 & 3a_0 + a_1 & 4a_0 + a_1 & \cdots \\ a_2 & a_1 + a_2 & a_0 + 2a_1 + a_2 & 3a_0 + 3a_1 + a_2 & 6a_0 + 4a_1 + a_2 & \cdots \\ a_3 & a_2 + a_3 & a_1 + 2a_2 + a_3 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$B^n a_k$: row $k+1$, column $n+1$ (first row and column indexed to 0)

Catalan Number Triangle

$$a_i = (1,\ -1, 0, 0, \dots), \qquad B^n a_i = \binom{n}{i} - \binom{n}{i-1}$$

$$\begin{bmatrix} \boxed{1} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ -1 & 0 & \boxed{1} & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 0 & -1 & -1 & 0 & \boxed{2} & 5 & 9 & 14 & 20 & \cdots \\ 0 & 0 & -1 & -2 & -2 & 0 & \boxed{5} & 14 & 28 & \cdots \\ 0 & 0 & 0 & -1 & -3 & -5 & -5 & 0 & \boxed{14} & \cdots \\ 0 & 0 & 0 & 0 & -1 & -4 & -9 & -14 & -14 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Right of zeros:

$$B^{2n} a_n = \frac{1}{n+1}\binom{2n}{n} = C_n$$

Discrete Convolution

Discrete convolution

If a_i and b_i are sequences, we define a new sequence, the **discrete convolution** (or Cauchy product) by

$$(a * b)_n = \sum_{i+j=n} a_i b_j = \sum_{i=0}^{n} a_i b_{n-i}.$$

Alternatively, $(a * b)_n$ is the coefficient c_n in the power series product

$$\sum_{i=0}^{\infty} c_i x^i = (\sum_{j=0}^{\infty} a_j x^j)(\sum_{k=0}^{\infty} b_k x^k).$$

Example: Catalan Numbers

$$C_0 = 1, \qquad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

Implement as iterated discrete convolution:

$$\begin{aligned}
C_1 &= 1 \cdot 1 = 1, \\
C_2 &= (1,1) \cdot (1,1) = 2, \\
C_3 &= (1,1,2) \cdot (2,1,1) = 5, \\
C_4 &= (1,1,2,5) \cdot (5,2,1,1) = 14, \\
C_5 &= (1,1,2,5,14) \cdot (14,5,2,1,1) = 42, \ldots.
\end{aligned}$$

Closed formula:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Frankel's Theorem (1950)

Theorem (Dwyer, Frankel)

If a_i and b_j are sequences, then, for all n in $\mathbb{Z}$,

$$(a * b)_k = (B^n a * B^{-n} b)_k.$$

Interpretation if $a_i = b_i$

1. construct array for $B^n a_k$,
2. section off any rectangle from the top row,
3. the convolution of the left and right hand columns is unchanged under telescoping.

Application: See Shapiro (1976)

Theorem (D.)

The length-squared of the n-th column of the Catalan triangle is $2C_n$.

Proof:

$$\begin{bmatrix} \boxed{1} & 1 & 1 & \boxed{1} & 1 & 1 & 1 & 1 & 1 & \cdots \\ \boxed{-1} & 0 & 1 & \boxed{2} & 3 & 4 & 5 & 6 & 7 & \cdots \\ \boxed{0} & -1 & -1 & \boxed{0} & 2 & 5 & 9 & 14 & 20 & \cdots \\ \boxed{0} & 0 & -1 & \boxed{-2} & -2 & 0 & \boxed{5} & 14 & 28 & \cdots \\ \boxed{0} & 0 & 0 & \boxed{-1} & -3 & -5 & \boxed{-5} & 0 & 14 & \cdots \\ 0 & 0 & 0 & 0 & -1 & -4 & -9 & -14 & -14 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

□

Proof of Frankel's Theorem:

Theorem (Dwyer, Frankel)

If a_i and b_j are sequences, then, for all n in $\mathbb{Z}$,

$$(a * b)_k = (B^n a * B^{-n} b)_k.$$

Suppose $p(x)$ and $q(x)$ correspond to a_i and b_j, respectively.

Define $[x^k]p(x) = a_k$. Then $(a * b)_k = [x^k](p(x)q(x))$.

$$\begin{aligned}(B^n a * B^{-n} b)_k &= [x^k]((1+x)^n p(x)(1+x)^{-n} q(x)) \\ &= [x^k](p(x)q(x)) \\ &= (a * b)_k. \qquad \square\end{aligned}$$

Topic: Magic Squares and Zeros of CGCs

Regge (1958):
domain space for CGCs corresponds precisely to 3x3 magic squares

$$V(m) \otimes V(n) \to V(m+n-2k)$$

$$M = \begin{bmatrix} n-k & m-k & k \\ i & j & * \\ m-i & n-j & i+j-k \end{bmatrix} \mapsto c_{m,n,k}(i,j)$$

Magic number: $J = m+n-k$

Recall: $f^i\phi_m \otimes f^j\phi_n$ pairs with $f^{m-i}\phi_m \otimes f^{n-j}\phi_n$.

$\mathbb{M}_3 =$ 3×3 weakly semi-magic squares

$$V(b+k) \otimes V(a+k) \to V(a+b)$$

$$M = \begin{bmatrix} a & b & k \\ r & * & * \\ * & * & c \end{bmatrix}$$

1. all entries are nonnegative integers,
2. all line sums along rows and columns are equal, and
3. this sum, the **magic number**, equals $J = a + b + k$.

Also called **integer "doubly-stochastic" matrices**.

Clebsch-Gordan coefficients/ function

$$C : \mathbb{M}_3 \to \mathbb{Z}$$

$$M = \begin{bmatrix} a & b & k \\ r & m & * \\ * & * & c \end{bmatrix} \mapsto$$

$$C(M) = \sum_{l=0}^{k} (-1)^l \binom{c}{r-l} \binom{b+k-l}{b} \binom{a+l}{a}$$

$C(M)$ is the coefficient of

$$x^m y^r$$

in the power series expansion of

$$\frac{(x+y)^c}{(1-x)^{b+1}(1+y)^{a+1}}$$

Determinantal Symmetries

For 3x3 matrices, let G be the group of determinantal symmetries;

that is,

1. G is generated by row switches, column switches, and transpose,
2. every element g of G may be expressed uniquely as

$$g = R(\sigma)C(\tau)T^{\epsilon} \quad \text{with} \quad \sigma, \tau \in S_3,$$

   and
3. $|G| = 72$.

These symmetries preserve

1. the semi-magic square property,
2. the magic number J, and
3. the zero locus for CGCs. (Regge, 1958)

Classification of Zeros:

Open problem: Classify the zeros of $C(M)$.
(Biedenharn, Brudno, Louck, K. S. Rao, etc.)

How to organize $\begin{bmatrix} a & b & k \\ r & * & * \\ * & * & c \end{bmatrix}$ as a subset of $\mathbb{N}^5$?

Three steps:

1. Fix a magic number $J \geq 0$,
2. partition by top-lines $\rightarrow$ equilateral triangle of size $J+1$, and
3. each top-line corresponds to a hexagon (as seen before).

That is, the data attached to

$$V(b+k) \otimes V(a+k) \rightarrow V(a+b)$$

is indexed by all magic squares with top line

$$(a, b, k).$$

Example: Hexagon without CGCs

$$\mathbf{J} = \mathbf{4}, \qquad \text{top line: } (1,\ 2,\ 1) \quad \to \quad 10 \text{ squares}$$

$$V(3) \otimes V(2) \to V(3)$$

$$\begin{bmatrix} 1 & 2 & 1 \\ r & * & * \\ * & * & c \end{bmatrix} \mapsto \begin{bmatrix} * & * & & \\ * & * & * & \\ & * & \boxed{*} & * \\ & & * & * \\ & & & \end{bmatrix}$$

Box corresponds to $(r, c) = (2, 2)$:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Note: No magic square for $(r, c) = (2, 0)$ if $J = 4$.

Example: Triangle of Top Lines

$$\mathbf{J = 4}$$

$$(0,0,4),(0,1,3),(0,2,2),(0,3,1),(0,4,0)$$

$$(1,0,3),(1,1,2),\boxed{(1,\ 2,\ 1),}(1,3,0)$$

$$(2,0,2),(2,1,1),(2,2,0)$$

$$(3,0,1),(3,1,0)$$

$$(4,0,0)$$

Top line entries: (rows above, spaces to left, spaces to right)

Example: Triangle with Magic Square Counts

$$\mathbf{J = 4}$$

$$
\begin{array}{c}
5,\quad 8,\quad 9,\quad 8,\quad 5 \\
8,\ 10,\ \boxed{10,}\ 8 \\
9,\ 10,\ 9 \\
8,\ 8 \\
5
\end{array}
$$

Total squares $= 120$

Example: $J = 8$

Magic Squares

$$
\begin{array}{ccccccccc}
\mathbf{9} & \mathbf{16} & \mathbf{21} & \mathbf{24} & \mathbf{25} & \mathbf{24} & \mathbf{21} & \mathbf{16} & \mathbf{9} \\
\mathbf{16} & \mathbf{22} & \mathbf{26} & \mathbf{28} & \mathbf{28} & \mathbf{26} & \mathbf{22} & \mathbf{16} \\
\mathbf{21} & \mathbf{26} & \mathbf{29} & \mathbf{30} & \mathbf{29} & \boxed{\mathbf{26}} & \mathbf{21} \\
\mathbf{24} & \mathbf{28} & \mathbf{30} & \mathbf{30} & \mathbf{28} & \mathbf{24} \\
\mathbf{25} & \mathbf{28} & \mathbf{29} & \mathbf{28} & \mathbf{25} \\
\mathbf{24} & \mathbf{26} & \mathbf{26} & \mathbf{24} \\
\mathbf{21} & \mathbf{22} & \mathbf{21} \\
\mathbf{16} & \mathbf{16} \\
\mathbf{9}
\end{array}
$$

CGC Zeros

$$
\begin{array}{ccccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \cdot & \mathbf{2} & \cdot & \cdot & \mathbf{2} & \cdot & \circ \\
\circ & \mathbf{2} & \mathbf{2} & \cdot & \mathbf{2} & \boxed{\mathbf{2}} & \circ \\
\circ & \cdot & \cdot & \cdot & \cdot & \circ \\
\circ & \cdot & \mathbf{2} & \cdot & \circ \\
\circ & \mathbf{2} & \mathbf{2} & \circ \\
\circ & \cdot & \circ \\
\circ & \circ \\
\circ
\end{array}
$$

Top line for box: 2 above, 5 to the left, 1 to the right $\mapsto (2, 5, 1)$

Example: $J = 8: \quad V(6) \otimes V(3) \to V(7)$

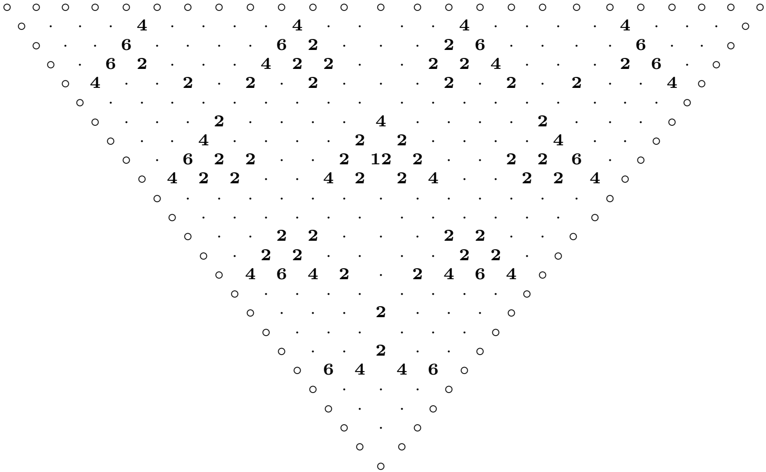
Top line $(2,5,1)$: 26 squares, 2 zeros

$$\begin{bmatrix} 6 & 6 & 6 & & & & & \\ -3 & 3 & 9 & 15 & & & & \\ & -3 & \boxed{0} & 9 & 24 & & & \\ & & -3 & -3 & 6 & 30 & & \\ & & & -3 & -6 & \boxed{0} & 30 & \\ & & & & -3 & -9 & -9 & 21 \\ & & & & & -3 & -12 & -21 \end{bmatrix}$$

Zeros: $(r,c) \;=\; (2,2),\ (4,5)$

$$\begin{bmatrix} 2 & 5 & 1 \\ 2 & 1 & 5 \\ 4 & 2 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 5 & 1 \\ 4 & 2 & 2 \\ 2 & 1 & 5 \end{bmatrix}$$

Example: $J = 24$: 252 Zeros, 6 Orbits



Example: $J = 24$: $V(16) \otimes V(16) \to V(16)$

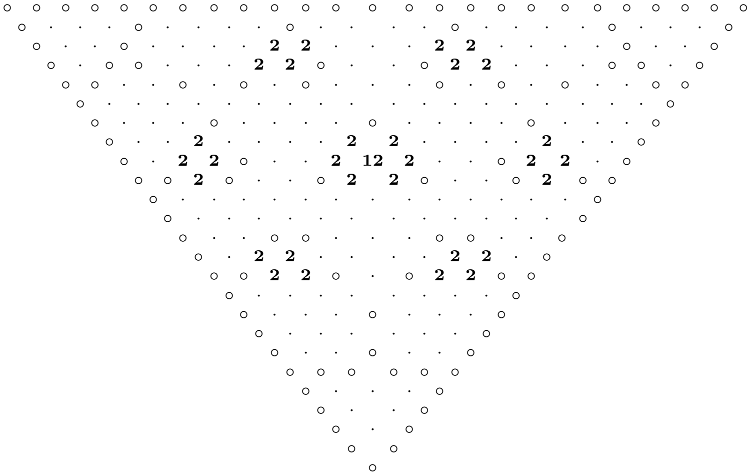
Top Line $(8, 8, 8)$, 12 zeros

12870	12870	12870	12870	12870	12870	12870	12870	12870								
-57915	-45045	-32175	-19305	-6435	6435	19305	32175	45045	57915							
135135	77220	32175	0	-19305	-25740	-19305	0	32175	77220	135135						
-212355	-77220	0	32175	32175	12870	-12870	-32175	-32175	0	77220	212355					
245025	32670	-44550	-44550	-12375	19800	32670	19800	-12375	-44550	-44550	32670	245025				
-212355	32670	65340	20790	-23760	-36135	-16335	16335	36135	23760	-20790	-65340	-32670	212355			
135135	-77220	-44550	20790	41580	17820	-18315	-34650	-18315	17820	41580	20790	-44550	-77220	135135		
-57915	77220	0	-44550	-23760	17820	35640	17325	-17325	-35640	-17820	23760	44550	0	-77220	57915	
12870	-45045	32175	32175	-12375	-36135	-18315	17325	34650	17325	-18315	-36135	-12375	32175	32175	-45045	12870
	12870	-32175	0	32175	19800	-16335	-34650	-17325	17325	34650	16335	-19800	-32175	0	32175	-12870
		12870	-19305	-19305	12870	32670	16335	-18315	-35640	-18315	16335	32670	12870	-19305	-19305	12870
			12870	-6435	-25740	-12870	19800	36135	17820	-17820	-36135	-19800	12870	25740	6435	-12870
				12870	6435	-19305	-32175	-12375	23760	41580	23760	-12375	-32175	-19305	6435	12870
					12870	19305	0	-32175	-44550	-20790	20790	44550	32175	0	-19305	-12870
						12870	32175	32175	0	-44550	-65340	-44550	0	32175	32175	12870
							12870	45045	77220	77220	32670	-32670	-77220	-77220	-45045	-12870
								12870	57915	135135	212355	245025	212355	135135	57915	12870

Zeros are orbits under column switches, R23 switch of

$$M = \begin{bmatrix} 8 & 8 & 8 \\ 2 & 9 & 13 \\ 14 & 7 & 3 \end{bmatrix}$$

Example: $J = 24$: Full orbit of 72 Zeros



Thank you!