Algebraic aspects of magic matrices and semi-magic squares

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Preamble:

Stony Brook Ph.D. written qualifier preparation question (1990s)

1. Find the characteristic and minimal polynomials of the following matrix U. Find bases for the eigenspaces.

$$U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

2. Repeat for the matrix

$$U' = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Solution to 1:

$$RREF: \qquad U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1. $\lambda = 0$: $\quad (1, -1, 0, 0), \quad (1, 0, -1, 0), \quad (1, 0, 0, -1)$
2. $\lambda = 4$: $\quad (1, 1, 1, 1)$

$$U^2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}^2 = 4U$$

1. $m_U(x) = x(x-4)$
2. $p_U(x) = x^3(x-4)$

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Joint work with M. S. Ravi

Definitions

For clarity, we only consider matrices with coefficients in $\mathbb{N}$ or $\mathbb{C}$

Definition

We say M in $M(n, \mathbb{C})$ is **row stochastic** with line sum r
if the sum along any row is r.

Likewise, define the notion of **column stochastic** with line sum c.

Magic matrices

Definition

M in $M(n, \mathbb{C})$ is called a **magic matrix** with line sum r if the sum along every row or column is r.

Define $MM(n)$ to be the set of all magic matrices of size n..

Note: If M is

- row stochastic with line sum r and
- column stochastic with line sum c

then $r = c$.

Proof: $nr = nc$ implies $r = c$.

Variations:

1. **Semi-magic squares** $\mathbb{M}(n)$: coefficients in $\mathbb{N}$
2. **Doubly stochastic**: coefficients in $0 \leq x \leq 1$, $r = c = 1$.

Example: Permutation matrices

Let $P(n)$ be the group of $n \times n$ matrices with entries

- exactly one 1 in each row and column, and
- 0 otherwise.

- $P(n) \cong S_n$ and $|P(n)| = n!$
- $P^T P = PP^T = I$
- $det(P) = \pm 1$
- magic matrix with line sum 1, and
- if $M = \sum x_i P_i$ then M is a magic matrix with line sum $\sum x_i$.

Birkhoff (1946): Polytope of DS matrices equals the convex hull of $P(n)$.

Example: Circulant matrices

Let $Z(n)$ be the subgroup of $n \times n$ matrices in $P(n)$ with entries

- all 1 along some "diagonal" to the right, and
- 0 otherwise.

Suppose $R = (123 \ldots n)$ is the element whose "diagonal" starts in the second entry of the first column. Then R generates all elements of $Z(n)$.

Of course, $R^n = I$ and $Z(n) \cong \mathbb{Z}/n$.

Example: $R = (123)$, $R^2 = (132)$, $R^3 = I$ in $Z(3)$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example: Circulant matrices

Circulant matrices

Let $C(n)$ be the commutative algebra generated by R in $Z(n)$.

That is, elements of $C(n)$ are linear combinations of the linearly independent matrices $I, R, R^2, \ldots R^{n-1}$.

$$C(3) = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}.$$

Basic Counting Problem: With coefficients in $\mathbb{N}$, how many elements of $C(n)$ have line sum L?

Solution: Identify $c_0 I + c_1 R + \cdots + c_{n-1} R^{n-1}$ with $(c_0, c_1, \ldots, c_{n-1})$.

Place L balls into n distinct boxes, giving $\binom{L+n-1}{L}$ squares.

Three approaches:

- combinatorics/ combinatorial number theory
  (counting the size of $\mathbb{M}(n, r)$)

  McMahon (1916); Stanley; DeLoera, and many others

- linear algebraic approaches
  ($MM(n)$ as a Lie algebra/Jordan algebra)
  Boukas, Feinsilver, Fellouris (2015)

- Our approach: the group algebra $\mathbb{C}[G]$
  1. Wedderburn's Theorem for semi-simple algebras over $\mathbb{C}$
  2. group actions.

Linear Algebra - Vector Spaces

If M_i is in $MM(n)$ with line sum L_i then

- $M_1 + M_2$ is row stochastic with line sum $L_1 + L_2$, and
- cM_1 is row stochastic with line sum cL_i.

So $MM(n)$ is a vector space over $\mathbb{C}$.

Linear Algebra - Dimensions

$$dim\ MM(n) = (n-1)^2 + 1^2$$

$$Example : dim\ MM(3) = 4 + 1 = 5$$

$$\begin{bmatrix} a & b & L-a-b \\ c & d & L-c-d \\ L-a-c & L-b-d & a+b+c+d-L \end{bmatrix}$$

Note:

$$(L-a-b) + (L-c-d) = (L-a-c) + (L-b-d)$$

Note:

$$(n-1)^2 + 1 \leq n! \text{ for } n \geq 1.$$

$P(n)$ spans $MM(n)$, but is not a basis. (More later)

Eigenvector Formulation

Let $u_1 = (1, 1, \ldots, 1)^T$ in $\mathbb{C}^n$. (column vector)

Alternative formulation

M is **row stochastic** with line sum L if and only if $Mu_1 = Lu_1$.
That is, u_1 is an eigenvector of M with eigenvalue L.

Alternative formulation

M is **column stochastic** with line sum L if and only if $M^T u_1 = Lu_1$.
That is, u_1 is an eigenvector of M^T with eigenvalue L.

Alternative formulation

M is a **magic matrix** with line sum L if and only if

$$Mu_1 = M^T u_1 = Lu_1.$$

That is, u_1 is an eigenvector of both M and M^T with eigenvalue L.

Multiplication

Proposition

Suppose M_i are row stochastic with line sums L_i.
Then $M_1 M_2$ is also row stochastic with line sum $L_1 L_2$.

Proof: $M_1 M_2 u_1 = M_1 L_2 u_1 = L_2 M_1 u_1 = L_1 L_2 u_1$. QED

Note that if M_i are instead column stochastic, then $(M_1 M_2)^T = M_2^T M_1^T$ is row stochastic with line sum $L_1 L_2$.

Conclusions:

- the product of two magic matrices is also magic,
- the line sum map $M \mapsto L_M$ is a linear character $L : MM(n) \to \mathbb{C}$, and
- if H is a subgroup of $P(n)$, then the algebra generated by H is a subalgebra of $MM(n)$.

Wedderburn's Theorem

Definition

If H is a subgroup of $P(n) \cong S_n$, then
define $MM_H(n)$ to be the algebra generated by H in $MM(n)$.

Wedderburn's Theorem

If A is a semisimple algebra over $\mathbb{C}$ of finite dimension, then

$$A \cong \bigoplus_i M(n_i, \mathbb{C}).$$

Consequences:

1. **Interpret:** there exists a basis such the elements of A are represented simultaneously by block diagonal matrices,
2. **Main Problem 1:** identify the block sizes n_i.
3. **Main Problem 2:** identify the orthogonal idempotents of A.

Group Algebras

Assume H is a subgroup of $P(n) \cong S_n$.

The group algebra of H

Define $\mathbb{C}[H]$ to be the vector space with basis $\{e_h\}_{h \in H}$.
Define multiplication in $\mathbb{C}[H]$ by extending $e_g \cdot e_h = e_{gh}$.

Of course, $\dim \mathbb{C}[H] = |H|$.

Consider the map of algebras, extending

$$\Phi : \mathbb{C}[H] \to MM_H(n) \subset MM(n)$$

$$\Phi(e_h) = h.$$

Example: $H = Z(n)$: Linear independence of $\{I, R, \ldots, R^{n-1}\}$

$$\Phi : \mathbb{C}[Z(n)] \xrightarrow{\sim} C(n)$$

Orthogonal idempotents for $C(n)$

One strategy: $Z(n)$ is cyclic (abelian)
$\rightarrow$ simultaneously diagonalize to get OIs.

Net effect: representation theory (more features)

Definition

A **character** of $Z(n)$ is a group homomorphism $\chi : Z(n) \rightarrow \mathbb{C}^*$.

Example (all): Let $\omega = e^{2\pi i/n}$. Fix $0 \leq k < n$. Then

$$\chi_k(R) = \omega^k$$

is a character of $Z(n)$.

Note:

$$|\chi(x)| = 1, \qquad \overline{\chi_k} = \chi_{-k}, \qquad \chi_k \cdot \chi_{k'} = \chi_{k+k'}$$

Orthogonal Relations for $Z(n)$

Orthogonality Relations for $Z(n)$

$$\frac{1}{|H|}\sum_{h\in H}\chi_k(h)\,\overline{\chi_{k'}(h)} = \begin{cases} 1 & k = k' \\ 0 & otherwise \end{cases}$$

Proof: If $k = k'$, clear.
If not,

$$\chi_k(h)\,\overline{\chi_{k'}(h)} = \chi_{k-k'}(h),$$

a non-trivial character.
So consider instead some χ with $\chi(x) \neq 1$. Set $S = \sum_h \chi(h)$.

$$\chi(x)S = \chi(x)\sum_{h\in H}\chi(h) = \sum_{h\in H}\chi(xh) = \sum_{h'\in H}\chi(h') = S.$$

Since $\chi(x) \neq 1$, $S = 0$. □

Representations of $Z(n)$

Let V be a finite-dimensional vector space over $\mathbb{C}$, and
define $GL(V)$ to be the set of invertible linear transformations $V \to V$.

Representation

A representation π of $Z(n)$ is a group homomorphism $\pi : Z(n) \to GL(V)$.

"Group action by linear transformations"

Full Reducibility to Characters

Every representation of $Z(n)$ may be diagonalized;
that is, there exists a basis such that

$$\pi(h) = \begin{bmatrix} \chi_(h) & 0 & 0 \\ 0 & \chi_2(h) & 0 \\ 0 & 0 & \chi_3(h) \end{bmatrix}.$$

Projection formula $P_\chi : MM(n) \to MM(n)_\chi$

$$P_\chi(v) = \frac{1}{|H|}\sum_{h\in H}\overline{\chi(h)}\;\pi(h)v \quad \to \quad P_\chi(M) = \frac{1}{n}\sum_{h\in Z(n)}\overline{\chi(h)}\;hM$$

Examples: $n=3,\ M=I \quad \to \quad$ Orthogonal idempotents

$$\chi_0(R)=1:\ 1\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+1\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}+1\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

$$\chi_1(R)=\omega:\ 1\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\omega^2\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}+\omega\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}=\begin{bmatrix}1&\omega&\omega^2\\\omega^2&1&\omega\\\omega&\omega^2&1\end{bmatrix}$$

$$\chi_2(R)=\omega^2:\ 1\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\omega\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}+\omega^2\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}=\begin{bmatrix}1&\omega^2&\omega\\\omega&1&\omega^2\\\omega^2&\omega&1\end{bmatrix}$$

Orthogonal Idempotents for $C(3)$

$$U_0 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad U_1 = \frac{1}{3}\begin{bmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{bmatrix}, \quad U_2 = \frac{1}{3}\begin{bmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{bmatrix}$$

Using $1 + \omega + \omega^2 = 0$,

Orthogonal Idempotents $\qquad P_i(M) = U_i M$

$$U_i^2 = U_i, \quad U_i U_j = 0 \ (i \neq j)$$

$$U_0 + U_1 + U_2 = I$$

$$C(3) = \mathbb{C}U_0 \oplus \mathbb{C}U_1 \oplus \mathbb{C}U_2 \cong \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

General $Z(n)$: Top line is $\chi_k(R^i) = \omega^{ik} \qquad (0 \leq i < n)$

Group actions for General Case:

There are two group actions of H on the vector spaces $\mathbb{C}[H]$ and $MM(n)$, respected by Φ; for instance, for all g in H,

$$\mathcal{L}(g)\Phi = \Phi\mathcal{L}(g)$$

$$\mathcal{L}(g)e_h = e_{gh} \quad \mapsto \quad \mathcal{L}(g)M = gM$$

$$\mathcal{R}(g)e_h = e_{hg^{-1}} \quad \mapsto \quad \mathcal{R}(g)M = Mg^{-1}$$

Example: In $C(3)$, we have the further identities

$$\mathcal{L}(R)U_0 = U_0 = \chi_0(R)U_0.$$

$$\mathcal{L}(R)U_1 = \omega U_1 = \chi_1(R)U_1.$$

$$\mathcal{L}(R)U_2 = \omega^2 U_2 = \chi_2(R)U_2.$$

That is, these orthogonal idempotents provide the basis that diagonalizes the group action in this case.

Projection formula (reprise):

Suppose $\mathcal{L}$ is in diagonal form already,

$$3P_\chi = \sum_h \overline{\chi(h)}\mathcal{L}(h) = \sum_h \overline{\chi(h)} \begin{bmatrix} \chi(h) & 0 & 0 \\ 0 & \chi_2(h) & 0 \\ 0 & 0 & \chi_3(h) \end{bmatrix}$$

$$= \sum_h \begin{bmatrix} \chi(h)\overline{\chi(h)} & 0 & 0 \\ 0 & \chi_2(h)\overline{\chi(h)} & 0 \\ 0 & 0 & \chi_3(h)\overline{\chi(h)} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Examples: $H = P(n) \cong S_n$

1. Φ has a large kernel $(n! > (n-1)^2 + 1)$ but is surjective,
2. the image has two blocks $(1^2 + (n-1)^2)$, and
3. the orthogonal idempotents are relatively easy: $U^2 = nU$

$$e_1 = \frac{1}{n}U, \qquad e_2 = I - \frac{1}{n}U$$

Orthogonal Idempotents

$$e_1 + e_2 = I, \qquad e_1 \cdot e_2 = 0, \qquad e_i^2 = e_i$$

4. Elements of $MM(n)$ with $L = 0$ have dimension $(n-1)^2$.

$$MM(n) = \mathbb{C}U \oplus \{L = 0\} \cong \begin{bmatrix} L & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

Example: $H = P(3) \cong S_3$

If $H = S_3$, then Φ is surjective but not injective.
By linear algebra, $P(3)$ is a linearly dependent set with dependence relation

$$I + R + R^2 - C - CR - CR^2 = 0.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$dim\ \mathbb{C}[S_3] = 6 = 1 + 5 = dim\ Ker\ \Phi + dim\ MM(3)$$

Basic Counting Problem in $MM(3)$ (1916)

To count elements with coefficients in $\mathbb{N}$ with line sum L,
we are considering elements of $\mathbb{C}[H]$ mod the dependence relation, or

$$\sum a_h e_h \mapsto (a_e, a_R, a_{R^2}, a_C, a_{CR}, a_{CR^2}) \in \mathbb{N}^6$$

mod the relation $(1, 1, 1, 0, 0, 0) = (0, 0, 0, 1, 1, 1)$.

Each element is now represented uniquely by a 6-tuple with sum L and such that a 0 occurs in the one of the last three entries.

$$H_3(L) = \binom{L+6-1}{L} - \binom{(L-3)+6-1}{L-3}.$$

First term: L balls into 6 boxes
Second term: $L-3$ balls into 6 boxes of this type $(0, 0, 0, 1, 1, 1)$

Example: Characters of S_3

The (one-dimensional) characters of S_3 are

1. the trivial character $\chi = 1$ (contributes U)
2. determinant as $P(3)$, or *sgn* as permutation.

Visually

$$\Phi : \begin{bmatrix} * & & & \\ & * & & \\ & & * & * \\ & & * & * \end{bmatrix} \to \begin{bmatrix} * & & \\ & * & * \\ & * & * \end{bmatrix}$$

In this case, the kernel is the idempotent in $\mathbb{C}[S_3]$ associated to *sgn*:

$$e_I + e_R + e_{R^2} - e_C - e_{CR} - e_{CR^2}$$

Characters for Non-abelian Groups

To access the 2×2 block:
Machinery from before still holds if we define the character of the representation π by

General character

$$\chi_\pi(h) = Trace(\pi(h)).$$

This is a function on H, but not a homomorphism in general.

Example: For the 2-dimensional representation of S_3 on $\mathbb{C}^2$ (induced from triangle in $\mathbb{R}^2$)

$$\chi_\pi(I) = 2, \qquad \chi_\pi(CR^i) = 0, \qquad \chi_\pi(R^i) = -1$$

Projections for Non-abelian Groups

We apply the same projection formula to get orthogonal idempotents in $MM(3)$:

$$U_{triv} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad U_{\pi} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$U_{triv}^2 = U_{triv}, \qquad U_{\pi}^2 = U_{\pi}, \qquad U_{triv}U_{\pi} = 0$$
$$U_{triv} + U_{\pi} = I$$

The four dimensional space is given by $U_{\pi}MM(3) = MM(3)U_{\pi}$, or $L = 0$ as seen before.

Example: Dihedral groups

Let D_{2n} be the union of all elements of $Z(n)$ and the corresponding matrices with "diagonals" to the left.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

With matrix multiplication, this set is isomorphic to the symmetry group of the regular n-gon. ($P = P^T$ if and only if $P^2 = I$.)

Magic matrices associated to D_{2n}

Let $D(n)$ be the non-commutative algebra generated by D_{2n} in $MM(n)$. That is, elements of $D(n)$ are linear combinations of the $2n$-elements

$$R^j, \quad CR^k \quad (1 \leq j, k \leq n)$$

Basic Counting Problem: DeLoera *et al* (2013)

Example: D_8

Consider

$$D(4) = MM_{D_8}(4) \subset MM(4),$$

where $D_8 \subset S_4$ is realized as symmetries of the square.

In this case, we shall see

$$\Phi : \begin{bmatrix} * & & & & & \\ & * & & & & \\ & & * & & & \\ & & & * & & \\ & & & & * & * \\ & & & & * & * \end{bmatrix} \to \begin{bmatrix} * & & & \\ & * & & \\ & & * & * \\ & & * & * \end{bmatrix}$$

Conjugacy classes and $\mathbb{C}[D_8]$

Conjugacy Classes

The center of $\mathbb{C}[H]$ is spanned by sums over conjugacy classes.
Thus the number of blocks equals the number of conjugacy classes.

Proof: Let C_x be the conjugacy class for x in H.

Then $\sum_{h \in C_x} e_h$ is clearly in the center of $\mathbb{C}[H]$.

Conversely, if $\sum c_x e_x$ is in the center, then

$$\sum c_x e_x = e_g (\sum c_x e_x) e_{g^{-1}} = \sum c_{g^{-1} x g} e_x.$$

Thus c_x is constant on conjugacy classes of H. $\square$

Conjugacy Classes and Blocks

Conjugacy classes for D_8:

1. e
2. R^2
3. $\{R, R^3\}$,
4. $\{C, CR^2\}$
5. $\{CR, CR^3\}$

First, this verifies the block count, noting D_8 is non-abelian:

$$8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$$

Character Table for D_8

Next, since Φ is surjective onto $D(4)$, it carries the center of $\mathbb{C}[D_8]$ into the center of $D(4)$.

D_8	e	R^2	R, R^3	$C,\ CR^2$	$CR,\ CR^3$
χ_{triv}	1	1	1	1	1
χ_{det}	1	1	1	-1	-1
χ_{sgn}	1	1	-1	-1	1
$\chi_{sgn} \cdot \chi_{det}$	1	1	-1	1	-1
π_2	2	-2	0	0	0

The characters yield the following matrices under P_χ:

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},\ 0,\ \frac{1}{4}\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix},\ 0,\ \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Orthogonal Idempotents

One verifies $U^2 = U, \quad U_1 U_2 = 0$, and

$$U_{triv} + U_{sgn} + U_{\pi_2} = I.$$

In particular, the latter item implies that $D(4)$ has three blocks and dimension 6.

Relations for counting problem

Note: There are two relations from the kernel of Φ:

$$I + R + R^2 + R^3 - C - CR - CR^2 - CR^3 = 0$$

$$I - R + R^2 - R^3 + C - CR + CR^2 - CR^3 = 0$$

or

$$I + R^2 = C(R + R^3) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$R + R^3 = C(I + R^2) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

General D_{2n} and $D(n)$

n odd:

1. 2 one-dimensional characters;
2. one relation for counting from *det*,
3. one dimensional block corresponds to U (all 1s), else size 2
4. Generating function for counting is

$$\frac{1-x^n}{(1-x)^{2n}}$$

$n = 2m$ even:

1. 4 one-dimensional characters;
2. two relations for counting (checkerboards),
3. two one-dimensional blocks in $D(n)$, else size 2
4. Generating function for counting is

$$\frac{(1-x^m)^2}{(1-x)^{2n}}$$