An intertwining operator for dihedral groups

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Joint work with M. S. Ravi

Motivation

$V(m)$ irreducible $SU(2)$-module over $\mathbb{C}$

Clebsch-Gordan Decomposition (Vector Space Level)

$$V(m) \otimes V(n) \cong V(|m-n|) \oplus \cdots \oplus V(m+n-2) \oplus V(m+n)$$

Choosing a basis without normalizing (no radicals):

Clebsch-Gordan Coefficients (Vector Level)

$$c_{m,n,k}(i,j) = \sum_{s=0}^{k} (-1)^s \binom{i+j-k}{i-s} \binom{m-s}{k-s} \binom{n-k+s}{s}$$

Semi-magic Squares

Observations by Regge (1950s):

Domain of $c_{m,n,k}(i,j)$

The domain space for Clebsch-Gordan coefficients may be parametrized by the set of semi-magic squares of size three.

Regge Symmetries

The symmetry group of these matrices has order 72:
generated by row/column switches and transpose.

Clebsch-Gordan coefficents transform well under these symmetries.

In normalized picture, scale by factor of $(-1)^N$ for some N.

Motivating Questions:

1. How much of the theory is **tensor products** and how much is **combinatorics**?

That is, are tensor products in this case an application of a purely combinatorial theory?

2. If so, is there a corresponding "Clebsch-Gordan coefficient" theory for general finite G?

Not in the sense of tensors, but permutation polytopes and semi-magic squares.

First step: need to understand how semi-magic matrices/squares work. So much of this talk is surveying.

Semi-Magic Matrices and Semi-Magic Squares

Definition

M in $M(n, \mathbb{C})$ is called a **semi-magic matrix** with line sum L if the sum along every row or column is L.

Define $MM(n)$ to be the set of all semi-magic matrices of size n.

Variations:

1. **Semi-magic squares** $\mathbb{M}(n)$: coefficients in $\mathbb{N}$
2. **Doubly stochastic**: coefficients in $0 \leq x \leq 1, \quad L = 1.$

Example: Permutation matrices

Let $P(n)$ be the group of $n \times n$ matrices with entries

- exactly one 1 in each row and column, and
- 0 otherwise.

- $P(n) \cong S_n$ and $|P(n)| = n!$
- $P^T P = PP^T = I$
- $det(P) = \pm 1$
- semi-magic matrix with line sum 1, and
- if $M = \sum x_i P_i$ then M is a semi-magic matrix with line sum $\sum x_i$.

Birkhoff (1946): Polytope of DS matrices equals the convex hull of $P(n)$.

Example: Circulant matrices

Let $Z(n)$ be the subgroup of $n \times n$ matrices in $P(n)$ with entries

- all 1 along some "diagonal" to the right, and
- 0 otherwise.

Suppose $R = (123 \ldots n)$ is the element whose "diagonal" starts in the second entry of the first column. Then R generates all elements of $Z(n)$.

Of course, $R^n = I$ and $Z(n) \cong \mathbb{Z}/n$.

Example: $R = (1234)$ in $Z(4)$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example: Circulant matrices

Circulant matrices

Let $Circ(n)$ be the commutative algebra generated by R in $Z(n)$.

That is, elements of $Circ(n)$ are **linear combinations** of the linearly independent matrices $I, R, R^2, \ldots R^{n-1}$.

$$Circ(3) = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_3 & c_1 & c_2 \\ c_2 & c_3 & c_1 \end{bmatrix}, \qquad L = c_1 + c_2 + c_3$$

Basic Counting Problem: With coefficients in $\mathbb{N}$,
how many elements of $Circ(n)$ have line sum L?

Solution: Identify $c_0 I + c_1 R + \cdots + c_{n-1} R^{n-1}$ with $(c_0, c_1, \ldots, c_{n-1})$.

Place L balls into n distinct boxes, giving $\binom{L+n-1}{L}$ squares.

Combinatorial Observations

Binomial coefficient as polynomial in L of degree $n-1$

$$H_n(L) = \binom{L+n-1}{L} = \frac{(L+n-1)(L+n-2)\ldots(L+1)}{(n-1)!}$$

- Generating Function: Binomial series

$$\sum_{L \geq 0} H_n(L)\, z^L \quad = \quad \frac{1}{(1-z)^n}$$

- Combinatorial Reciprocity:

$$H_n(-L) \quad = \quad (-1)^{n-1}\, H_n(L-n)$$

and

$$H_n(-1) \;=\; H_n(-2) \;=\; \ldots \;=\; H_n(-n+1) = 0.$$

Three approaches to semi-magic matrices/squares:

- combinatorics/ combinatorial number theory
  (counting the size of $\mathbb{M}(n)$ with fixed line sum L)

  McMahon (1916); Stanley; DeLoera, and many others

- linear algebraic approaches
  ($MM(n)$ as a Lie algebra/Jordan algebra)
  Boukas, Feinsilver, Fellouris (2015)

- Our approach: the group algebra $\mathbb{C}[G]$
  1. Wedderburn's Theorem for semi-simple algebras over $\mathbb{C}$
  2. group actions.

Linear Algebra - Vector Spaces

If M_i is in $MM(n)$ with line sum L_i then

- $M_1 + M_2$ is semi-magic with line sum $L_1 + L_2$, and
- cM_1 is semi-magic with line sum cL_i.

So $MM(n)$ is a vector space over $\mathbb{C}$.

Dimension of $MM(n)$

$$dim\ MM(n) = (n-1)^2 + 1^2$$

$P(n)$ spans $MM(n)$, but is not a basis.

$$(n-1)^2 + 1 < n! \qquad \text{for} \qquad n \geq 3.$$

Linear Algebra - Dimensions

Dimension of $MM(n)$

$$dim\ MM(n) = (n-1)^2 + 1^2$$

For the subspace with $L = 0$ (actually a simple ideal), a basis is given by the $(n-1)^2$ linearly independent vectors:

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 1 \end{array}\right]$$

For the extra dimension, we can use

- identity I_n with $L = 1$, or
- J_n (all 1s) with $L = n$.

Eigenvector Formulation

Let $u_1 = (1, 1, \ldots, 1)^T$ in $\mathbb{C}^n$. (column vector)

Alternative formulation

M is **row stochastic** with line sum L if and only if $Mu_1 = Lu_1$.
That is, u_1 is an eigenvector of M with eigenvalue L.

Alternative formulation

M is **column stochastic** with line sum L if and only if $M^T u_1 = Lu_1$.
That is, u_1 is an eigenvector of M^T with eigenvalue L.

Alternative formulation

M is a **semi-magic matrix** with line sum L if and only if

$$Mu_1 = M^T u_1 = Lu_1.$$

That is, u_1 is an eigenvector of both M and M^T with eigenvalue L.

Multiplication

Proposition

Suppose M_i are row stochastic with line sums L_i.
Then $M_1 M_2$ is also row stochastic with line sum $L_1 L_2$.

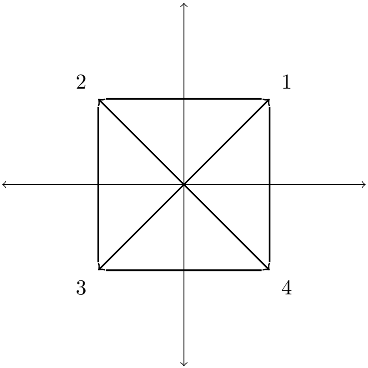
Proof: $M_1 M_2 u_1 = M_1 L_2 u_1 = L_2 M_1 u_1 = L_1 L_2 u_1$. QED

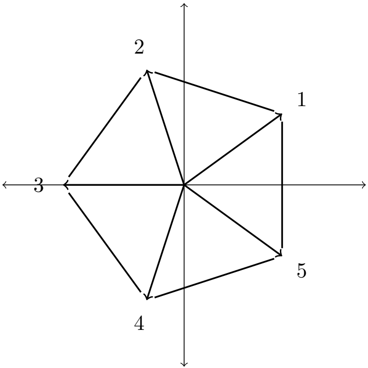
Note that if M_i are instead column stochastic, then $(M_1 M_2)^T = M_2^T M_1^T$ is row stochastic with line sum $L_1 L_2$.

Conclusions:

- the product of two semi-magic matrices is also semi-magic,
- the line sum map $M \mapsto L_M$ is a linear character $L : MM(n) \to \mathbb{C}$, and
- if G is a subgroup of $P(n)$,
  then products of linear combinations of G are LCs of G.

Main Example: $G = D_{2n}$





Vertices and orientation for D_8 and D_{10}

Four Pictures for D_{2n}

- symmetries of regular polygon with n sides,
- as a subgroup of S_n: permutations of the vertices,

$$R = (12...n), \quad C = (1n)(2\ n-1)\dots$$

- as a subgroup of permutation matrices, and
- finite presentation:

$$|R| = n, \qquad |C| = 2, \qquad CRC = R^{-1}.$$

With $0 \leq k < n$, every element is of the form

- R^k (rotation), or
- CR^k (reflection).

$Z(n) \subset D_{2n} \subseteq S_n$ as $P(n)$

The element C is chosen as reflection across the x-axis.

- Multiply by C on left: invert columns, and
- multiply by C on right: invert rows.

With this choice, all reflections CR^k are (-1)-circulant.
That is, constant along diagonals to the left.

$$\underset{C}{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}, \underset{CR}{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}, \underset{CR^2}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}, \underset{CR^3}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}.$$

Two Themes for the Remainder:

$$D_{2n} \;=\; Z(n) \;\cup\; CZ(n)$$

$$\mathbb{M}(D_{2n}) \;=\; Span_{\mathbb{N}}(D_{2n}) \qquad \text{(monoid)}$$

$$MM(D_{2n}) \;=\; Span_{\mathbb{C}}(D_{2n}) \qquad \text{(algebra)}$$

- Formula for counting elements in $\mathbb{M}(D_{2n})$ with fixed line sum L.
- Basic structure of $MM(D_{2n})$ as an extension of $Circ(n)$.

Group Algebras

Assume G is a subgroup of $P(n) \cong S_n$.

The group algebra of G

Define $\mathbb{C}[G]$ to be the vector space with basis $\{e_h\}_{h \in G}$.
Define multiplication in $\mathbb{C}[G]$ by extending $e_g \cdot e_h = e_{gh}$.

Of course, $\dim \mathbb{C}[G] = |G|$.

Consider the map of algebras, extending

$$\Phi : \mathbb{C}[G] \to MM(G) \subset MM(n)$$

$$\Phi(e_h) = h.$$

Example: $G = Z(n)$: Linear independence of $\{I, R, \ldots, R^{n-1}\}$

$$\Phi : \mathbb{C}[Z(n)] \xrightarrow{\sim} Circ(n)$$

Example: $G = D_6 = S_3 \cong P(3)$

If $G = D_6$, then Φ is surjective but not injective to $MM(3)$.
By linear algebra, $P(3)$ is a linearly dependent set with dependence relation

$$I + R + R^2 - C - CR - CR^2 = 0.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$dim\ \mathbb{C}[S_3] = 6 = 1 + 5 = dim\ Ker\ \Phi + dim\ MM(3)$$

Counting: n odd

Define

$$\chi_{det} : D_{2n} \to \mathbb{C}^*, \qquad \chi_{det}(P) = \det(P).$$

Then $Ker(\Phi)$ is the span of the element in $\mathbb{C}[D_{2n}]$:

$$\sum_{P \in D_{2n}} \chi_{det}(P)\ e_P.$$

In plain language, this says, in $MM(D_{2n})$,

$$\sum \text{rotations} = \sum \text{reflections}$$

or

$$\sum \text{circulant} = \sum (-1)\text{-circulant}.$$

Counting: n odd

Model: order rotations before reflections as P_i.

Then we may consider semi-magic squares in $\mathbb{M}(D_{2n})$ as $2n$-tuples of natural numbers

$$\sum c_i P_i \quad \mapsto \quad (c_1, \ldots, c_n, c_{n+1}, \ldots, c_{2n})$$

subject to the relation

$$v \; + \; (1,1,\ldots,1,0,0,\ldots,0) \; = \; v \; + \; (0,0,\ldots,0,1,1,\ldots,1).$$

Then each semi-magic square is uniquely represented by $2n$-tuple with at least one zero in the first n entries.

Counting: n odd

Theorem (D.-Ravi)

Suppose n is odd.
The number of semi-magic squares in $\mathbb{M}(D_{2n})$ with line sum L equals

$$H_{D_{2n}}(L) = \binom{L+2n-1}{L} - \binom{L+n-1}{L}.$$

The corresponding generating function is

$$F_n(x) = \sum_{L \geq 0} H_{D_{2n}}(L)\, x^L \; = \; \frac{1-x^n}{(1-x)^{2n}}$$

Proof: Distribute L balls to $2n$ boxes for first term.
For uniqueness, remove all $2n$-tuples of the form

$$(1, 1, \ldots, 1, 0, 0, \ldots, 0) + (c_1, \ldots, c_{2n}). \quad \square$$

Counting: $n = 2m$ even

We now have four characters: $triv$, det, sgn, $sgn \cdot det$

Now $Ker(\Phi)$ is the span of two elements in $\mathbb{C}[D_{2n}]$:

$$\sum_{P \in D_{2n}} \chi(P)\ e_P,$$

where χ is *det* and another non-trivial character (parity of m).

In plain language, these reduce to four groupings,

$$\sum R^{2k} = \sum CR^{2k+1}$$

and

$$\sum R^{2k+1} = \sum CR^{2k}.$$

Counting: $n = 2m$ even

Checkerboards with $n = 4$:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = C \cdot \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = C \cdot \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Counting: $n = 2m$ even

Theorem (D.-Ravi)

Suppose $n = 2m$.
The number of semi-magic squares in $\mathbb{M}(D_{2n})$ with line sum L has generating function

$$F_n(x) = \frac{(1 - x^m)^2}{(1 - x)^{2n}}.$$

Proof: $4m$-tuples, 4 segments

$$(c_1, \ldots, c_m \mid c_{m+1}, \ldots, c_{2m} \,||\, c_{2m+1}, \ldots, c_{3m} \mid c_{3m+1}, \ldots, c_{4m})$$

Uniqueness: move 1s between segments 1 and 2 only, 3 and 4 only.
Then

$$H_{D_{2n}}(L) = \sum_{k=0}^{L} h_n(k) h_n(L-k),$$

where $h_n(L)$ is the formula for the odd case, but allowing n even.

Discrete convolution → multiply generating functions. □

Combinatorial Reciprocity

Other counting formulas for D_{2n} using polytope theory:
Burggraf et al. (2013), Baumeister et al. (2014)

Corollary

- For n odd (resp. even),

  $H_{D_{2n}}(L)$ is a polynomial in L with degree $2n-2$ (resp. $2n-3$),
- $H_{D_{2n}}(-L) = (-1)^{n-1} H_{D_{2n}}(L-n)$, and
- $H_{D_{2n}}(-1) = H_{D_{2n}}(-2) = \cdots = H_{D_{2n}}(-n+1) = 0.$

This result is the D_{2n}-analogue of the Anand-Dumir-Gupta Conjecture, proven by Stanley in the 1970s.

$G = S_n$: Explicit polynomials for $\mathbb{M}(n)$ are known only up to $n = 9$.

Algebras: Intertwining Operator Φ

On $\mathbb{C}[G]$, $G \times G$ acts by

$$(h_1, h_2) \cdot e_P = e_{h_1 P h_2^{-1}}.$$

On $MM(G)$, $G \times G$ acts by

$$(h_1, h_2) \cdot P = h_1 P h_2^{-1}.$$

So Φ intertwines the $G \times G$-actions:

$$(h_1, h_2) \cdot \Phi(e_P) = h_1 P h_2^{-1} = \Phi((h_1, h_2) \cdot e_P).$$

Algebras: Intertwining Operator Φ

With π ranging over a complete set of irreducibles,

$$\mathbb{C}[G] \;\cong\; \bigoplus_{\pi} V_\pi \otimes V_\pi^*$$

as representations of $G \times G$. So

$$\mathbb{C}[G] \;\cong\; Ker(\Phi) \oplus MM(G).$$

Multiplicity one means the items on the left are determined by checking if π occurs in the defining permutation representation ρ.

Basic Case: Orthogonal idempotents for $Circ(n)$

Here Φ is an isomorphism.

Describe $Im(\Phi) = Circ(n)$ via $Z(n) \times Z(n)$.

$Z(n)$ is cyclic (abelian)
$\rightarrow$ simultaneously diagonalize to get OIs.

All irreducibles are characters.
Let $\omega = e^{2\pi i/n}$. Fix $0 \leq k < n$. Then

$$\chi_k(R) = \omega^k$$

is a character of $Z(n)$.

Projection formula $P_\chi : Circ(n) \to Circ(n)_\chi$

$$P_\chi(v) = \frac{1}{|G|} \sum_{h \in G} \overline{\chi(h)}\ \pi(h) v \quad \to \quad P_\chi(M) = \frac{1}{n} \sum_{h \in Z(n)} \overline{\chi(h)}\ hM$$

Examples: $n = 3,\ M = I \quad \to \quad$ Orthogonal idempotents

$$\chi_0(R) = 1 :\ 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\chi_1(R) = \omega :\ 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \omega^2 \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \omega \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{bmatrix}$$

$$\chi_2(R) = \omega^2 :\ 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \omega \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \omega^2 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{bmatrix}$$

Orthogonal Idempotents for $Circ(3)$

General: Top line is $\chi_k(R^i) = \omega^{ik} \qquad (0 \leq i < n)$

$$U_0 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad U_1 = \frac{1}{3}\begin{bmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{bmatrix}, \quad U_2 = \frac{1}{3}\begin{bmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{bmatrix}$$

Using $1 + \omega + \omega^2 = 0$,

Orthogonal Idempotents $\qquad P_i(M) = U_i M$

$$U_i^2 = U_i, \quad U_i U_j = 0 \ (i \neq j)$$

$$U_0 + U_1 + U_2 = I$$

$$Circ(3) = \mathbb{C}U_0 \oplus \mathbb{C}U_1 \oplus \mathbb{C}U_2 \cong \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

Examples: $G = P(n) \cong S_n$

1. Φ has a large kernel $(n! > (n-1)^2 + 1)$ but is surjective,
2. the permutation representation has two components $(1^2 + (n-1)^2)$, and
3. the orthogonal idempotents are relatively easy: $J =$ all 1s

$$JM = MJ = LJ, \quad J^2 = nJ \quad \rightarrow \quad e_1 = \frac{1}{n}J, \qquad e_2 = I - \frac{1}{n}J$$

Orthogonal Idempotents

$$e_1 + e_2 = I, \qquad e_1 \cdot e_2 = 0, \qquad e_i^2 = e_i$$

4. $\{L = 0\}$ is a simple ideal in $MM(n)$ with dimension $(n-1)^2$.

$$MM(n) = \mathbb{C}J \oplus \{L = 0\} \cong \begin{bmatrix} L & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

Example: D_{2n} (n odd)

Character table for D_{2n} (n odd)

D_{2n}	e	$R^{\pm 1}$	$R^{\pm 2}$	$\dots$	C
$\lvert C_\sigma \rvert$	1	2	2	$\dots$	n
χ_{triv}	1	1	1	$\dots$	1
χ_{det}	1	1	1	$\dots$	-1
π_2	2	$2\cos(\frac{2\pi}{n})$	$2\cos(\frac{4\pi}{n})$	$\dots$	0
ρ	n	0	0	$\dots$	1

There are $\frac{n-1}{2}$ conjugacy classes of type $R^{\pm k}$.

With $2 \leq j \leq \frac{n-1}{2}$ and homomorphisms

$$\phi_j : D_{2n} \to D_{2n} : \qquad R \mapsto R^j, \quad C \mapsto C,$$

Other characters $\to \pi_{2,j} = \pi_2 \circ \phi_j \ \to\ 2\cos(\frac{2j\pi}{n})$.

$Ker(\Phi)$ and $MM(D_{2n})$

By orthogonality of characters, as representations of $D_{2n} \times D_{2n}$,

- $Ker(\Phi)$ contains one constituent, of type $\chi_{det} \otimes \chi^*_{det}$,
- $MM(D_{2n})$ contains a trivial type and one for each $\pi_{2,j} \otimes \pi^*_{2,j}$

Define $c_t = \cos(\frac{2\pi t}{n})$. Then the orthogonal idempotent for $\pi_{2,j}$ is given by

$$U_{2,j} = P_{2,j}(I) = \frac{1}{2n} \sum_{k=0}^{n-1} 2c_{jk}\, R^k = \frac{1}{n} \begin{bmatrix} 1 & c_j & c_{2j} & c_{3j} & \dots \\ c_j & 1 & c_j & c_{2j} & \dots \\ c_{2j} & c_j & 1 & c_j & \dots \\ c_{3j} & c_{2j} & c_j & 1 & \dots \\ \dots & & & & \end{bmatrix}$$

Orthogonal Idempotents

$$U_{2,j} = \frac{1}{n}\begin{bmatrix} 1 & c_j & c_{2j} & c_{3j} & \dots \\ c_j & 1 & c_j & c_{2j} & \dots \\ c_{2j} & c_j & 1 & c_j & \dots \\ c_{3j} & c_{2j} & c_j & 1 & \dots \\ \dots & & & & \end{bmatrix}$$

Things worth noting:

1. with $\frac{1}{n}J$, the $U_{2,j}$ form a complete set of OIs for $MM(D_{2n})$,
2. each $U_{2,j}$ is symmetric and circulant (as a sum of circulants),
3. each $U_{2,j}$ is semi-magic with line sum 0,
4. each $U_{2,j}$ is the real part of an orthogonal idempotent for $Circ(n)$,
5. the imaginary parts are also interesting.

$$1 + \omega + \cdots + \omega^{n-1} = 0 \quad \rightarrow \quad 1 + \cos(2\pi/n) + \cdots + \cos(2\pi(n-1)/n) = 0$$

Thank you!