Lattice path enumeration for semi-magic squares of size three

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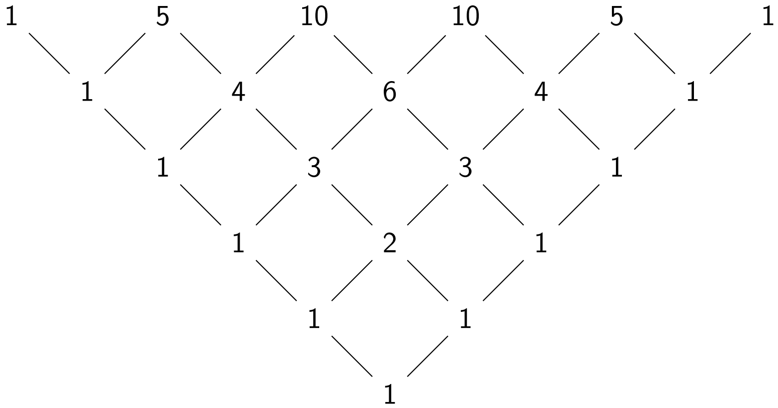
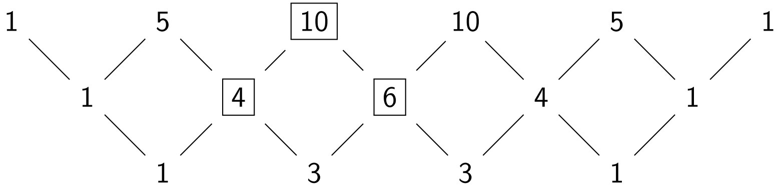
Pascal's Triangle



Interpret: lattice path counting from the vertex $\hat{0}$ to the entry

Pascal's Identity



Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Point: To get an entry in the *n*-th row, add the two below.
Lattice path counting → "up operator"

Partially Ordered Sets

Definition:

A non-empty set P with binary relation $\leqslant$ is called a

partially ordered set (poset)

if, for all x, y, z in P, the binary relation $\leqslant$ satisfies the following properties

1. Reflexive: $x \leqslant x$,
2. Anti-symmetric: if $x \leqslant y$ and $y \leqslant x$, then $x = y$, and
3. Transitive: if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$.

Finite Graded Posets

$(P, \leqslant)$ finite poset with $\hat{0}$ and $\hat{1}$ (minimum and maximum)

P **graded** of rank n:

The length of every path from $\hat{0}$ to $\hat{1}$ equals the same n

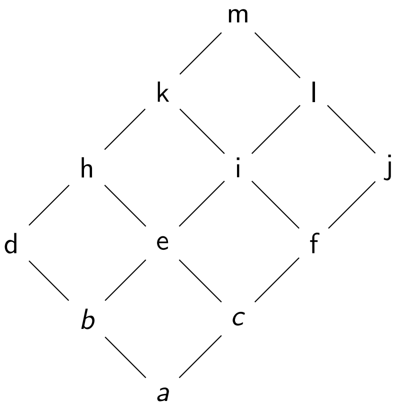
Rank function $\rho : \ P \to \{0, 1, \ldots, \ n\}$

$$\rho(x) = \text{length of any path from } \hat{0} \text{ to } x$$

Rank numbers

$$P_t = \{x \in P \mid \rho(x) = t\} \qquad\qquad |P_t| = p_t$$

Finite Graded Posets (Hasse Diagram of P)



$\hat{0} = a, \qquad \rho(h) = 3, \qquad P_3 = \{h, i, j\}, \qquad p_3 = 3$

The Vector Space for a Finite Graded Poset with $\hat{0}$ and $\hat{1}$

Use elements of P as a basis (any order)

Definition: The vector space $\mathbb{R}[P]$

Let $\mathbb{R}[P]$ be the vector space over $\mathbb{R}$ with formal basis P;

that is, elements of $\mathbb{R}[P]$ are linear combinations

$$v = \sum_{x \in P} c_x \, x$$

Definition: $x \lessdot y$

For x and y in P, we say y **covers** x if $x \leqslant y$ and no z satisfies $x < z < y$.

The Order-Raising Operator U

Definition: $U : \mathbb{R}[P] \to \mathbb{R}[P]$

For x in P, linearly extend the map

$$Ux = \sum_{x \lessdot y} y, \qquad U(\hat{1}) = 0$$

Note that Ux is the formal sum of all elements of P directly "above" x.

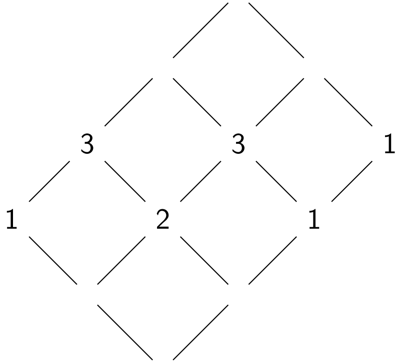
That is, $U|_{P_t} : \mathbb{R}[P_t] \to \mathbb{R}[P_{t+1}]$.

Alternatively, if y is in P_{t+1}, then the coefficient of y in

$$U(\sum_{x \in P} c_x \ x)$$

is the sum of all values c_x just "below" y.

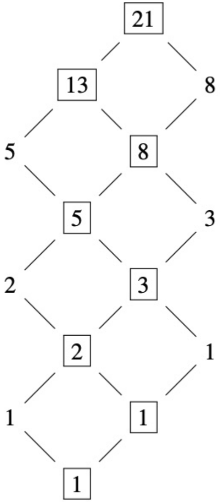
The Order-Raising Operator U

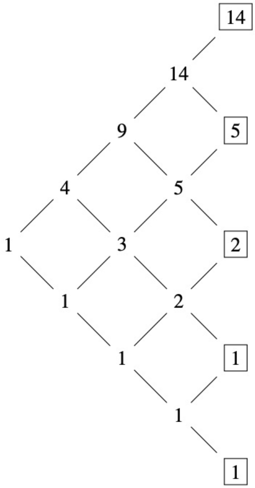


$$U : \mathbb{R}[P_2] \to \mathbb{R}[P_3]$$

$$U(x_1 + 2x_2 + x_3) = 1(x_4) + 2(x_4 + x_5) + 1(x_5 + x_6)$$
$$= 3x_4 + 3x_5 + x_6$$

Fibonacci Numbers / Catalan Numbers





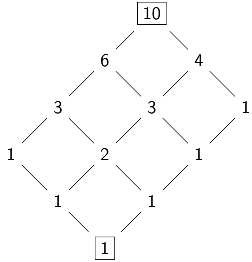
Clebsch-Gordan Decomposition for $m \times n$ grid

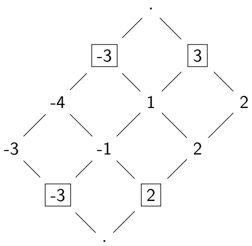
- U is nilpotent:
$$U^{m+n+1} = 0$$
- Jordan Canonical form: $\lambda = 0$
- On rank t, define
$$Hv = (m+n-2t)v$$
- H-eigenvalues on $Ker(U)$:
$$-|m-n|, \quad -|m-n|+2, \quad \ldots, \quad -m-n-2, \quad -m-n$$
- complete to $\mathfrak{sl}(2,\mathbb{R})$: there exists a $D|_{P_t} : \ P_t \to P_{t-1}$ s.t.
$$[D,U] = H, \qquad [H,D] = 2D, \qquad [H,U] = -2U$$

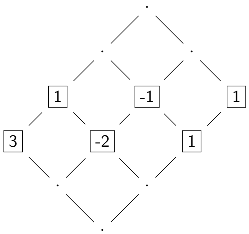
Hasse diagrams for $V(N)$ for 2×3 grid

$V(N)$: U-cyclic subspace corresponding to H-eigenvalue $-N$,

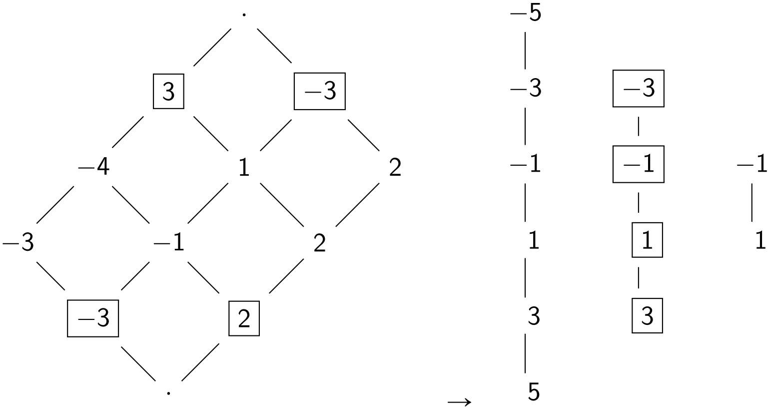
$V(5)$	$V(3)$	$V(1)$
$k = 0$	$k = 1$	$k = 2$







Weight vectors vs. Weights



Points in rank t: weak compositions (i,j) with 2 parts
Rank 3:

$$-4\ x_{(2,1)}\ +\ 1\ x_{(1,2)}\ +\ 2\ x_{(0,3)}$$

Five Parameters and Uniform Formula

Parameters:

$$M: \qquad m \times n, \qquad m+n-2k, \qquad (i,j)$$

Un-normalized Clebsch-Gordan coefficent

$$C(M) \; = \; \sum_t (-1)^t \binom{i+j-k}{i-t} \binom{m-t}{k-t} \binom{n-k+t}{t}.$$

Generating function

$C(M)$ is the coefficient of $x^j y^i$ in

$$\frac{(x+y)^{i+j-k}}{(1-x)^{m-k+1}(1+y)^{n-k+1}}$$

Clebsch-Gordan Coefficients

Peck poset: a f.g. poset with $\hat{0}$ and $\hat{1}$ and $\mathfrak{sl}(2,\mathbb{R})$ action.

(80s: Stanley, Proctor; later, also Robert G. Donnelly)

Rectangular grid: automatically carries action through tensor product:

$V(N)$: irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$ of highest weight $N \geqslant 0$
ϕ_N certain highest weight vector for $V(N)$

$C(M) = c_{m,n,k}(i,j)$

$$V(m+n-2k) \subseteq V(m) \otimes V(n),$$

$$f^{t-k}\ \phi_{m+n-2k} = \sum_{i+j=t} c_{m,n,k}(i,j)\ f^i \phi_m \otimes f^j \phi_n.$$

Summation Formulas

$$C(M) \;=\; \sum_t (-1)^t \binom{i+j-k}{i-t}\binom{m-t}{k-t}\binom{n-k+t}{t}.$$

Formulas of this type (normalized): Wigner, Majumdar, Racah
See, for instance, Vilenkin, "Special Functions...", 1968.

$$\rightarrow \quad M \;=\; \begin{bmatrix} n-k & m-k & k \\ i & j & m' \\ m-i & n-j & i+j-k \end{bmatrix}$$

M is **semi-magic** with line sum $m+n-k$

Semi-Magic Squares of Size 3

A square matrix is called a **semi-magic square** if

1. entries are integers $\geqslant 0$, and
2. the sum along any row or column is equal to the same number L.

L is called the **line sum** of M.

Example: Let $M(3)$ be the monoid of all semi-magic squares of size 3.

$$M = \begin{pmatrix} 3 & 2 & 4 \\ 5 & 3 & 1 \\ 1 & 4 & 4 \end{pmatrix}, \qquad L = 9.$$

Examples: $L = 1$

A **permutation matrix** is a square matrix such that there is exactly one 1 in each row and column.

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Algebra for Semi-Magic Squares

1) If $k \geqslant 0$ and M, N are semi-magic squares, so are

$$kM, \qquad M + N$$

with line sums kL_M and $L_M + L_N$, respectively.

2) Any linear combination with $a_i \geqslant 0$ integers

$$a_1 P_1 + \cdots + a_6 P_6$$

is a semi-magic square with line sum $a_1 + \cdots + a_6$.

3) Birkhoff-Von Neumann:
Semi-magic squares of any size may be written as an integral sum of permutation matrices.

Linearly Independent? No.

Solve:

$$\sum a_i P_i = \begin{pmatrix} a_1 + a_6 & a_3 + a_5 & a_2 + a_4 \\ a_2 + a_5 & a_1 + a_4 & a_3 + a_6 \\ a_3 + a_4 & a_2 + a_6 & a_1 + a_5 \end{pmatrix} = 0.$$

Dependence relation:

$$P_1 + P_2 + P_3 = P_4 + P_5 + P_6 = J,$$

or

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Rectangles

Every semi-magic square M can be represented by a rectangle:

$$M \quad \leftrightarrow \quad \mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6) \quad \leftrightarrow \quad \begin{array}{|c|c|c|} \hline a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ \hline \end{array}.$$

Here the single relation takes the form

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 1 & 1 & 1 \\ \hline \end{array}.$$

By repeatedly shifting up, uniquely represented if one of a_4, a_5, a_6 is zero.

Note: line sum $L = a_1 + \cdots + a_6$ is unchanged by shifting 1s

Counting by Line Sum

Question: How many semi-magic squares are there with fixed line sum L? (MacMahon 1916)

$$H_3(L) = \binom{L+5}{5} - \binom{L+2}{5}$$

First term: put L balls in 6 boxes.

Second term: put $L-3$ balls in 6 boxes

Throw away rectangles of the form: $(L-3)+3 = L$

$$\begin{array}{|c|c|c|}\hline a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 1 & 1 & 1 \\ \hline \end{array}.$$

Put L balls in k boxes? $L+(k-1)$ choose $k-1$
(assume 1s at ends) $L = 5, k = 4: \quad \rightarrow \quad 1: \ 01000101 \ :1$

Wreath Product $G = S_3 \wr \mathbb{Z}/2$

At matrix level, the magic square property and line sum are preserved by

1. row permutations,
2. column permutations, and
3. transpose.

At rectangle level, the effect is to

1. switch rows
2. allow permutations in row entries.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 2 \\ \hline 4 & 5 & 6 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 4 & 5 & 6 \\ \hline 1 & 2 & 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 6 & 4 & 5 \\ \hline 2 & 3 & 1 \\ \hline \end{array}, \quad \ldots$$

Lattice Path Counting: Graded Poset $M(3)$

$M(3)$ forms a graded poset:
Partial ordering (entry-wise for all entries):

$$M \leqslant N \quad \text{if} \quad m_{ij} \leqslant n_{ij}$$

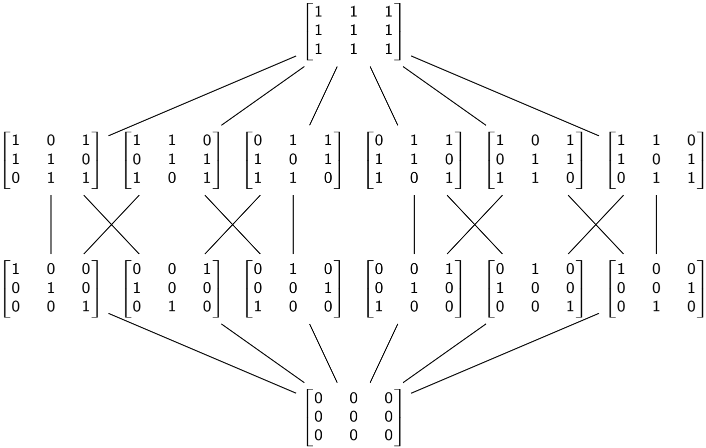
$$M \lessdot N \quad \text{if} \quad N = M + P_i \text{ for some } i$$

Rank function $\rho(M) = L = \sum a_i$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \leqslant \begin{pmatrix} 2 & 4 & 0 \\ 1 & 2 & 3 \\ 3 & 0 & 3 \end{pmatrix}$$

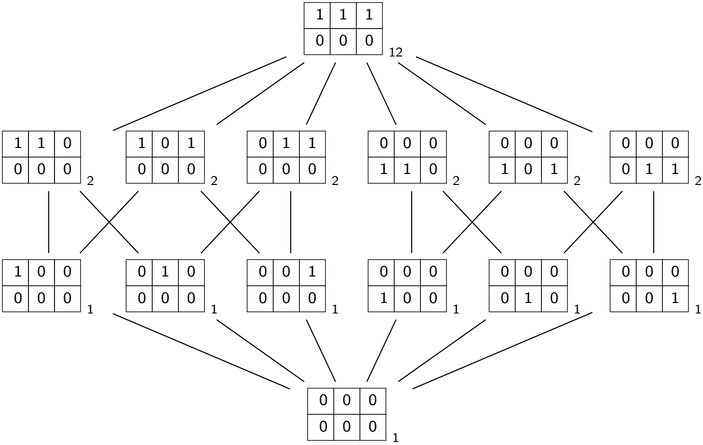
$$\rho(M) = 3 \quad \leqslant \quad 6 = \rho(N)$$

$M(3,1)$ as semi-magic squares: $\max(M) \leqslant 1$



$$M(3,s) \ = \ \{M \in M(3) \mid max(M) \leqslant s\}$$

$M(3,1)$ as rectangles:



Subscript: Number of directed paths from $\hat{0}$ to M

$M(3,2)$ as semi-magic squares: $\max(M) \leqslant 2$

Hasse diagram:

1. columns go from rank 0 to rank 6,
2. no covering links for clarity, and
3. only orbits denoted.

		$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix}$		
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$
		$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$		
0	1	2	3	4	5	6

$M(3,2)$ as rectangles: $\max(M) \leqslant 2$

Subscripts are:

1. number of elements in the orbit,
2. number of directed paths from $\hat{0}$ to M

$$\begin{array}{|c|c|c|}\hline 2&0&0\\\hline 0&0&0\\\hline\end{array}_{6,1} \quad \begin{array}{|c|c|c|}\hline 2&1&0\\\hline 0&0&0\\\hline\end{array}_{12,3} \quad \begin{array}{|c|c|c|}\hline 2&2&0\\\hline 0&0&0\\\hline\end{array}_{6,6}$$

$$\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&0&0\\\hline\end{array}_{1,1} \quad \begin{array}{|c|c|c|}\hline 1&0&0\\\hline 0&0&0\\\hline\end{array}_{6,1} \quad \begin{array}{|c|c|c|}\hline 1&1&0\\\hline 0&0&0\\\hline\end{array}_{6,2} \quad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 0&0&0\\\hline\end{array}_{1,12} \quad \begin{array}{|c|c|c|}\hline 2&1&1\\\hline 0&0&0\\\hline\end{array}_{6,36} \quad \begin{array}{|c|c|c|}\hline 2&2&1\\\hline 0&0&0\\\hline\end{array}_{6,150} \quad \begin{array}{|c|c|c|}\hline 2&2&2\\\hline 0&0&0\\\hline\end{array}_{1,900}$$

$$\begin{array}{|c|c|c|}\hline 1&0&0\\\hline 1&0&0\\\hline\end{array}_{9,2} \quad \begin{array}{|c|c|c|}\hline 1&1&0\\\hline 1&0&0\\\hline\end{array}_{18,6} \quad \begin{array}{|c|c|c|}\hline 1&1&0\\\hline 1&1&0\\\hline\end{array}_{9,24}$$

To get to $\begin{array}{|c|c|c|}\hline 3&3&3\\\hline 0&0&0\\\hline\end{array}$: 94,080 paths

To get to $\begin{array}{|c|c|c|}\hline 4&4&4\\\hline 0&0&0\\\hline\end{array}$: 11,988,900 paths $\rightarrow$ OEIS: A306642, A000172

How many lattice paths from $\hat{0}$ to M?

These are words of length $\rho(M)$ in $\{P_i\}$ that sum to M.

$$2J: \quad P_1P_1P_2P_3P_2P_3, \quad P_3P_4P_1P_6P_2P_5$$

There are

$$\binom{\rho(M)}{a_1, a_2, a_3, a_4, a_5, a_6}$$

directed paths from $\hat{0}$ to $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6)$ in $\mathbb{N}^6$.

If $m_0 = \min(M)$, then there are $m_0 + 1$ ways to represent M using the syzygy:

$$\begin{array}{|c|c|c|}\hline a_1 & a_2 & m_0 \\ \hline a_4 & a_5 & 0 \\ \hline \end{array} + t \begin{array}{|c|c|c|}\hline -1 & -1 & -1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

if one of a_1, a_2, a_3 equals m_0 and one of a_4, a_5, a_6 is zero.

Path Counting Formula

Theorem: (D.) Lattice path count from $\hat{0}$ to M

$$v(M) = \sum_{t=0}^{m_0} \binom{\rho(M)}{a_1 - t,\ a_2 - t,\ a_3 - t,\ a_4 + t,\ a_5 + t,\ a_6 + t}$$

$$v(J) = \binom{3}{1,\ 1,\ 1,\ 0,\ 0,\ 0} + \binom{3}{0,\ 0,\ 0,\ 1,\ 1,\ 1} = 6 + 6 = \boxed{12}$$

$$v(2J) = \binom{6}{2,\ 2,\ 2,\ 0,\ 0,\ 0} + \binom{6}{1,\ 1,\ 1,\ 1,\ 1,\ 1} + \binom{6}{0,\ 0,\ 0,\ 2,\ 2,\ 2}$$

$$= 90 + 720 + 90 = \boxed{900}$$

$$v(3J) = 1680 + 45360 + 45360 + 1680 = \boxed{94080}$$

$v(M)$ is evidently invariant under the action of G on $\begin{array}{|c|c|c|}\hline a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ \hline \end{array}$.

Clebsch-Gordan Coefficients

$v(M)$ has a factor of hypergeometric type ${}_3F_2$.
So do Clebsch-Gordan coefficients.

Definition

$$F(\mathbf{a}, z) = \sum_{t=0}^{m_0} \binom{\rho(M)}{a_1 - t,\ a_2 - t,\ a_3 - t,\ a_4 + t,\ a_5 + t,\ a_6 + t} z^t,$$

where $m_0 = \min(a_1, a_2, a_3)$ and at least one of a_4, a_5, a_6 equals 0.

Polynomial: insert z into definition of $v(M)$

Reciprocity?

Theorem ((D.) Reciprocity for Clebsch-Gordan coefficients)

Suppose $\mathbf{a}$ *is the representative for* M *in* $M(3)$ *with at least one of* a_4, a_5, a_6 *equal to 0 and* $m_0 = \min(a_1, a_2, a_3)$. *Then*

$$F(\mathbf{a}, 1) \;=\; v(M)$$

and

$$F(\mathbf{a}, -1) \;=\; (-1)^{a_2+m_0} \binom{\rho(M)}{a_1+a_5,\; a_2+a_6,\; a_3+a_4} C(M).$$

$F(\mathbf{a}, 1)$: maximal chains/facets

Combinatorial Reciprocity: Binomial coefficients

$$f(n) = \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!}$$

Choose k objects from n.

$$f(-n) = (-1)^k \binom{n+k-1}{k} = (-1)^k f(n+k-1)$$

Put k balls in n boxes

Recommended for $M(3)$:
R. Stanley's slides from 2014 NYU talk
Magic Squares and Syzygies (durer.pdf)

Application: The 72 Regge Symmetries

Effect of $G = S_3 \wr \mathbb{Z}/2$ on $C(M)$:

Traditional: Regge (1958)
Wigner 3-j symbol (CG coefficient with extra normalizing factor)
$\rightarrow$ G acts by sign changes based on parity

$C(M)$:
effect is complicated, but directly derived from theorem

Example: Consider switching columns 1 and 2 in

$$M = \begin{bmatrix} a_1 + a_6 & a_3 + a_5 & a_2 + a_4 \\ a_2 + a_5 & a_1 + a_4 & a_3 + a_6 \\ a_3 + a_4 & a_2 + a_6 & a_1 + a_5 \end{bmatrix}$$

The corresponding permutation is $(15)(24)(36)$ (fixes column 3)

Regge Symmetry: Switch Columns 1 and 2

Permutation: (15)(24)(36): Then

$$(a_1, a_2, a_3, a_4, a_5, a_6) \mapsto \mathbf{a}' = (a_5, a_4, a_6, a_2, a_1, a_3)$$
$$\mapsto \mathbf{a}'' = (a_5, a_4, a_6, a_2, a_1, a_3) + m_0(1,\ 1,\ 1, -1, -1, -1)$$

and

$$F(\mathbf{a}, -1) \ = \ (-1)^{m_0} F(\mathbf{a}'', -1)$$

or (multinomial terms cancel)

$$(-1)^{a_2+m_0} C(M) \ = \ (-1)^{a_4+3m_0} C(M'')$$

or

$$c_{m,n,k}(i,j) = \ (-1)^k c_{n,m,k}(j,i)$$

That is,

$$V(m) \otimes V(n) \to V(n) \otimes V(m)$$

Thank you!

References

1. P. MacMahon, Combinatory Analysis
2. R. Stanley, Enumerative Combinatorics, Volume 1
   (semi-magic squares, Ehrhart reciprocity)
3. R. Stanley, Algebraic Combinatorics
   (Peck posets, wreath products)