Methods of Validation for Adynkra Models in Supersymmetry

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Acknowledgements

Overall project is joint work with

1. S. J. Gates, Jr. (University of Maryland),
2. T. Hübsch (Howard University), and
3. R. Nath (York College-CUNY)

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Overview:

Main theme of the talk:
Graphical methods in supersymmetry and representation theory

Part 1: Adinkras

Combinatorics, hypercubes, and weight spaces

Part 2: Adynkras

Representation theory and irreducible representations of $\mathfrak{so}(n)$

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Part 1: Adinkras

Joint work with S. James Gates, Jr., Tristan Hübsch, and Rishi Nath

Preprint: A combinatorial introduction to Adinkras, arXiv:2410.12834

1. Physical Motivation

Example: Boson-Fermion Exchange in Supersymmetry

Supermultiplet in supersymmetry with 4 component fields

1. Bosons/fermions: $\{b_1, b_2, f_1, f_2\}$ as functions of time,
2. Supercharges: linear operators Q_1, Q_2 on functions,
3. Relations among the supercharges:

$$Q_1^2 \;=\; Q_2^2 \;=\; i\,\frac{d}{dt}, \quad Q_1 Q_2 = -Q_2 Q_1$$

- $Q_1(\boxed{b_1}) = \frac{d}{dt}\boxed{f_1}$
- $Q_1(\boxed{b_2}) = \boxed{f_2}$
- $Q_1(\boxed{f_1}) = i\,\boxed{b_1}$
- $Q_1(\boxed{f_2}) = i\frac{d}{dt}\boxed{b_2}$

- $Q_2(\boxed{b_1}) = -\frac{d}{dt}\boxed{f_2}$
- $Q_2(\boxed{b_2}) = \boxed{f_1}$
- $Q_2(\boxed{f_1}) = i\frac{d}{dt}\boxed{b_2}$
- $Q_2(\boxed{f_2}) = -i\,\boxed{b_1}$

Coupled Differential Equations/ Representation

Bosons/fermions: $(b_1(t), b_2(t), f_1(t), f_2(t))$ as column vector

Supercharges Q_1, Q_2 in these coordinates:

$$Q_1 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i\frac{d}{dt} \\ \frac{d}{dt} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad Q_2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i\frac{d}{dt} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{d}{dt} & 0 & 0 & 0 \end{bmatrix}$$

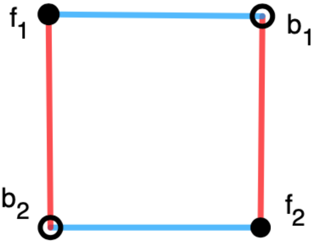
Check:

$$Q_1^2 = Q_2^2 = i\frac{d}{dt}, \qquad Q_1 Q_2 = -Q_2 Q_1$$

Drop $\frac{d}{dt}$, i, and signs: permutation matrices

Drop $\frac{d}{dt}$ and i: signed permutation matrices

Graphical Model



- Q_1 interchanges the vertices of the blue edges.
- Q_2 interchanges the vertices of the red edges.

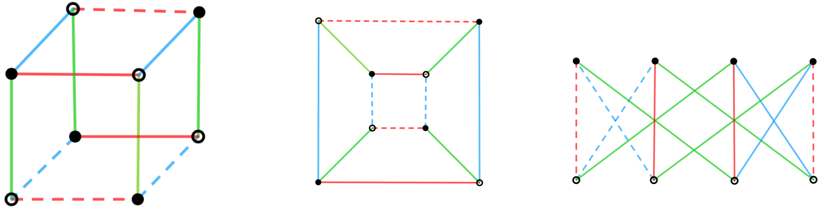
Key concept in general:

- Bipartition on vertices (bosons and fermions)
- Bijections between these two sets.
- Each supercharge induces a **perfect matching.**

Adinkra Model for $\{Q_i\}$

Augment similar bipartite graphs with

- 1-factorization (or regular edge coloring),
- the quadrilateral property,
- totally odd dashing, and
- a partial ordering.



Hypercube $\longrightarrow$ Graph $\longrightarrow$ Poset

2. Matchings on graphs

Let G be a finite graph with vertex set V and edge set E.

Definition

A **1-factor** (or **perfect matching**) of G is a disjoint subset of edges that meet all vertices.

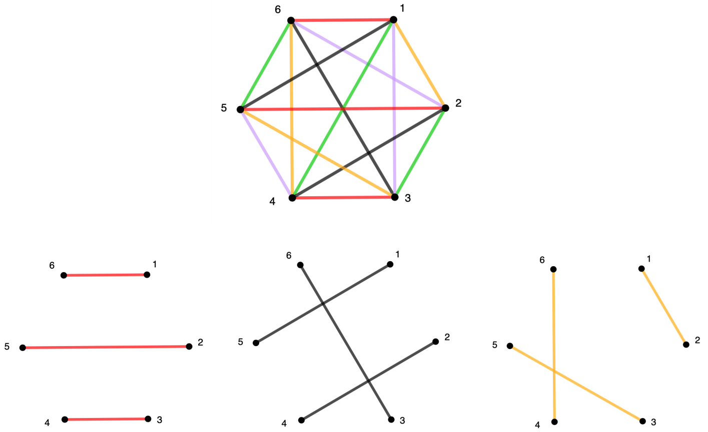
Definition

A **1-factorization** of G is a partition of the edge set into 1-factors.

The existence of a 1-factorization is equivalent to the existence of a **regular edge coloring**.

That is, there is a coloring of the edge set such that each color meets each vertex exactly once.

Example: Complete graphs of even order



Existence of perfect matching requires an even number of vertices.

1-factorizations

Corollary to Hall's Marriage Theorem

Let G be a finite graph. If G is bipartite and k-regular, then G admits a 1-factorization.

We recall some definitions:

Definition

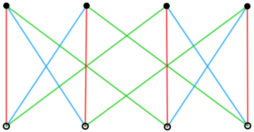
A finite graph G is called k-**regular** if each vertex has degree k.

1-factorization: k colors implies k-regular

Bipartite

Definition

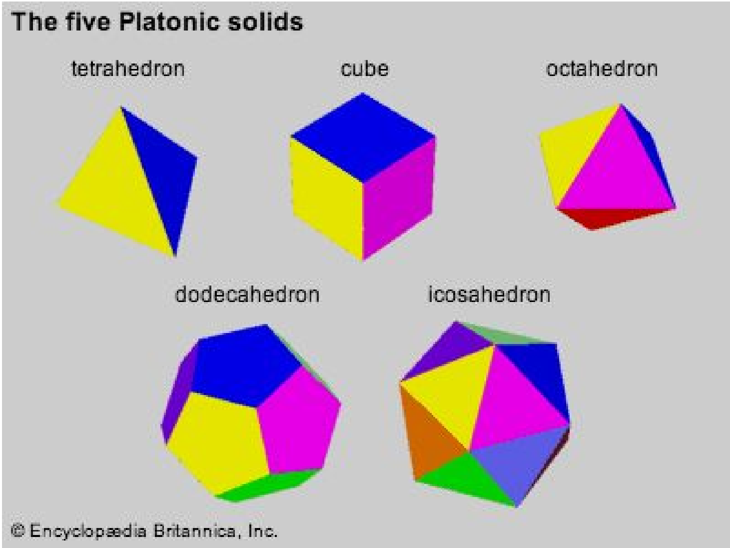
A graph G is called **bipartite** if the vertex set V of G partitions into subsets V_1 and V_2 such that each edge joins a vertex in V_1 to a vertex in V_2.



Theorem

Let G be a finite graph. G is bipartite if and only if G has no odd cycles.

Examples: Edge sets of Platonic solids



More 1-factorizations

Bipartite:

- Cube,
- Truncated octahedron,
- Great rhombicuboctahedron,
- Great rhombicosidodecahedron.

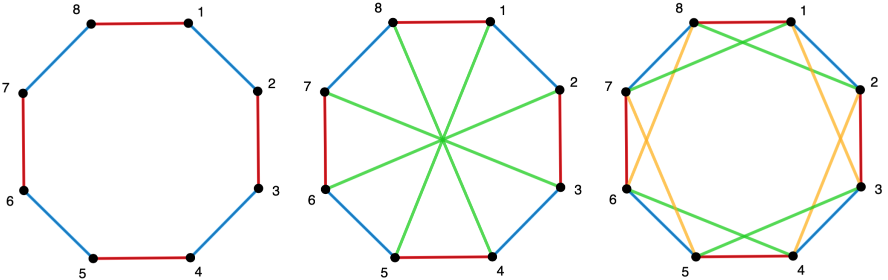
Each Platonic solid admits a 1-factorization.

Other places to look for 1-factorizations:

- regular polytopes in higher dimensions
- other permutohedra (Coxeter groups / finite reflection groups)
- hypercubes, quotients, and adinkras

3. The Quadrilateral Property and Hypercubes

Circulant graphs and involutions in S_8



Blue:	(12)(34)(56)(78)	(12)(34)(56)(78)	(12)(34)(56)(78)
Red:	(18)(23)(45)(67)	(18)(23)(45)(67)	(18)(23)(45)(67)
Green:		(15)(26)(37)(48)	(17)(28)(35)(46)
Orange:			(13)(24)(57)(68)

Context: Quotients of Coxeter groups

Definition

A group H is called a **Coxeter group** if

- it is generated by a finite subset $\{s_i\}$ of involutions ($s_i^2 = e$),
- for all $i \neq j$, $(s_i s_j)^{m_{ij}} = e$ for some positive integer $m_{ij} \geqslant 2$, and
- no other relations exist.

Since there may be extra relations, our finite group is a quotient of a Coxeter group. Interesting, but not used in what follows.

Question: When is the group associated to a 1-factorization **abelian**?
If s_i is the involution for color i, then

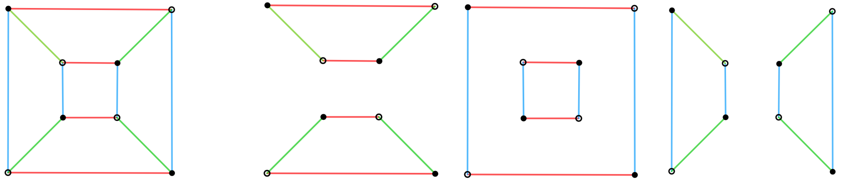
$$s_i^2 = e \qquad \text{implies} \qquad s_i = s_i^{-1}.$$

$$\text{So} \qquad s_i s_j = s_j s_i \qquad \text{is equivalent to} \qquad s_i s_j s_i s_j = e.$$

The Quadrilateral Property

Definition

A 1-factorization of a graph G has the **quadrilateral property** if, when restricting to any two colors, these edges form a disjoint set of 4-cycles.



The Quadrilateral Property

Proposition

The group H associated to a 1-factorization of a finite graph is abelian if and only if the quadrilateral property holds.

Consequence:

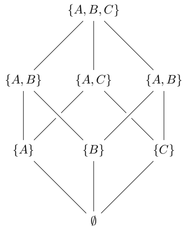
If G is connected, then G is a quotient of a hypercube

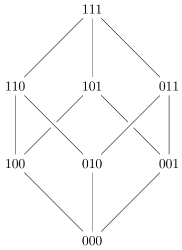
Edge set of a Hypercube

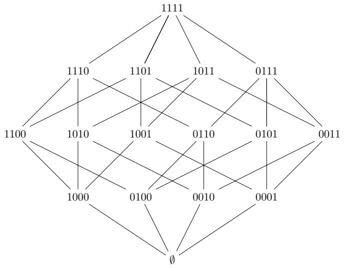
1. Boolean lattice as a power set
2. Linear binary block codes and Hamming distance
3. Direct product of $\mathbb{Z}/2$ groups

Distance between codes of same length = number of differing entries

Hasse diagram: Draw edges when distance is 1.



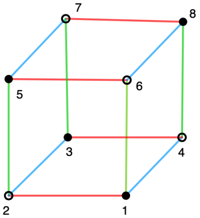




Parallel Edge Coloring of a Hypercube

- 1-factorization: use parallel edges for colorings
  For instance, join each $0yz$ to $1yz$ with same color
- Quadrilateral property: a complete set of squares follows from choosing any two colors

  Induction: removing one color splits the hypercube into two smaller cubes as in first item
- Associated group: product of $\mathbb{Z}/2$-groups

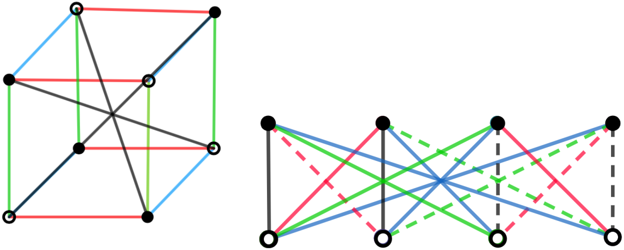


Blue:	(14)(23)(57)(68)
Green:	(16)(25)(37)(48)
Red:	(12)(34)(56)(78)

$$H = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$$

Non-Cubic Example of The Quadrilateral Property

For a non-cubic example, consider the complete bipartite graph $K_{4,4}$ (add 4 diagonals to the 3-cube example)



This example carries the smallest non-cubic adinkra:
the graph is 4-regular, but the 4-regular cube (the 4-cube) has 16 vertices.

4. Definition: Adinkra

An **adinkra** is a finite graph G with the following properties

1. Bipartite and k-regular (implies 1-factorization)
2. the quadrilateral property
3. Totally odd dashing
4. a compatible partial ordering

First defined in:
M. Faux and S. J. Gates, Jr., Adinkras: a graphical technology for supersymmetric representation theory, Phys. Rev. D (3) 71 (2005)

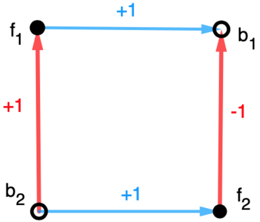
Supersymmetry Revisited

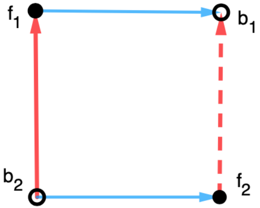
Recall that $Q_k^2 = i\frac{d}{dt}$, so that values of Q_k on vertices on an edge for color k must have the same sign.

Now include signs for signed graphs, but denote -1 by a dashed edge.

- $Q_1(\boxed{b_1}) = \frac{d}{dt}\boxed{f_1}$
- $Q_1(\boxed{b_2}) = \boxed{f_2}$
- $Q_1(\boxed{f_1}) = i\boxed{b_1}$
- $Q_1(\boxed{f_2}) = i\frac{d}{dt}\boxed{b_2}$

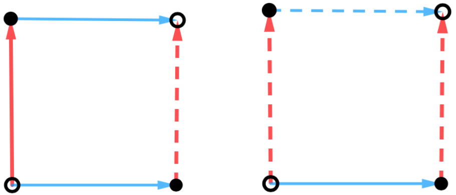
- $Q_2(\boxed{b_1}) = \frac{d}{dt}\boxed{-f_2}$
- $Q_2(\boxed{b_2}) = \boxed{f_1}$
- $Q_2(\boxed{f_1}) = i\frac{d}{dt}\boxed{b_2}$
- $Q_2(\boxed{f_2}) = i\boxed{-b_1}$





Totally odd dashing

To reflect the condition $Q_k Q_j = -Q_j Q_k$ for any number of supercharges, each 4-cycle from the quadrilateral property must have 1 or 3 dashed edges.



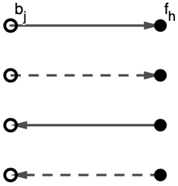
That is, the product of signs in the signed 4-cycle equals -1.

Arrows for partial order

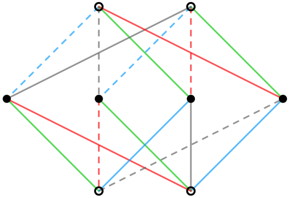
$Q_k^2 = i\frac{d}{dt}$

4 choices for Q_k if $Q_k(f_j)$ carries the i term:

- $Q_k(b_j) = \quad f_h$
- $Q_k(b_j) = -f_h$
- $Q_k(b_j) = \quad \frac{d}{dt} f_h$
- $Q_k(b_j) = -\frac{d}{dt} f_h$



Point arrows up for Hasse diagram:



Classification by codes

Realize the vertices of hypercube as the group $H = \mathbb{Z}/2 \times \ldots \times \mathbb{Z}/2$ via binary words of length n.

This is also a vector space over the field $\mathbb{Z}/2$.

Definition

The **weight** of a binary word is the number of 1s in the word. A binary word is called **even (resp. doubly even)** if its weight is even (resp. a multiple of four).

Definitions

A code is a subset of binary words in H. A code is called **even (resp. doubly even)** is all of its codewords are. A **linear code** is a vector subspace of H.

Proposition

A linear code is even (resp. doubly even) if and only if every codeword in a basis is even (resp. a multiple of four, and each bitwise "and" between basis elements has even weight). Bitwise "and" $=$ keep only matching 1s.

Examples of doubly even codes

$$d_4 : \qquad \{0000, 1111\} \qquad \text{(smallest non-cubical adinkra)}$$

Bases:

$$d_6 : \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$d_8 : \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$e_7 : \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Doubly even linear codes are classified up to length 28.
(See slide 29 reference (DFGHILM), Table 3.)

Classification by codes

Choose a linear code C in $H = (\mathbb{Z}/2)^n$ and form the quotient space. Edges and colors are determined by the action of weight 1 generators of H.

The following graph properties result if all codewords have the property:

Weight of codeword		Quotient $\mathbb{Z}_2{}^n/C$
even	$\leftrightarrow$	bipartite
$\neq 1, 2$	$\leftrightarrow$	quadrilateral
doubly even	$\leftrightarrow$	totally odd

Source for table: Kevin Iga, Adinkras: Graphs of Clifford algebra representations, supersymmetry, and codes, arXiv:2110.01665v1, 2021.

Classification by codes

The **chromotopology** of an adinkra is its underlying undirected bipartite graph with its undashed edge coloring.

Theorem (DFGHILM, 2011)

There is a one-one correspondence between possible chromotopologies of connected adinkras and doubly even linear block binary codes of length N.

Doran, Faux, Gates, Hübsch, Iga, Landweber, and Miller, Codes and supersymmetry in one dimension, Adv. in Th. Math. Phys. 15 (2011) 1909-1970.

Related mathematical topics

1. Signed permutation matrices and Clifford algebras
2. Riemann surfaces and cubic cohomology
3. Spin representations
4. Associahedra and cluster algebras

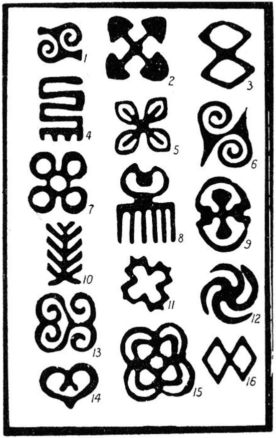
Part 2: Adynkras / Tensor Product Decompositions

Where to begin for mathematicians:

- Yan Zhang, Adinkras for mathematicians, Trans. Am. Math. Soc. 366 (6) (2014), 3325-3355.
- C. F. Doran, K. M. Iga, and G. D. Landweber, An application of cubical cohomology to adinkras and supersymmetry representations, Ann. Henri Poincaré Comb. Physics Interact. 4 (2017), 387-415.
- Kevin Iga, Adinkras: Graphs of Clifford algebra representations, supersymmetry, and codes, arXiv:2110.01665v1 (2021).
- K. Iga, C. Klivans, J. Kostiuk, and C. H. Yuen, Eigenvalues and critical groups of adinkras, Adv. in Appl. Math. 143 (2023) 102450.

Adinkra symbols (Ghana)

Recorded by Robert Rattray (1927):



Part 2: Adynkras

1. Adynkras as Hasse diagrams
2. Dominant weight sets for irreducible representations
3. Characters for exterior algebras
4. Validation and spreadsheet models (Excel)

Four Excel workbooks, preprint in progress

Motivating Problem: Adynkras

Particle Physics/ Supersymmetry:

1. Grassmann variables for superfields in quantum field theory,
2. Adynkrafields as an enhancement of superfields

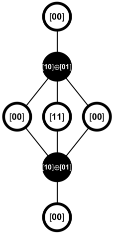
Purely mathematical description of Adynkras:

Representation Theory of the orthogonal Lie algebra $\mathfrak{so}(n)$:

Tensor product decompositions of an exterior algebra $\bigwedge^* S$, where S is a spin representation or sum of two spin representations.

Connection to Adinkras

Forgetful Functor:
Restriction from $\bigwedge^* S$ for $\mathfrak{so}(4)$ to its weight spaces



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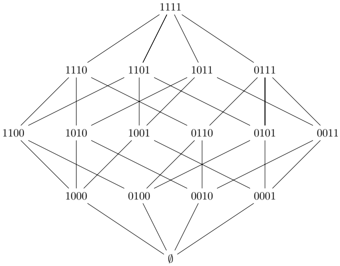


Some results of GHM (2020-21, 3 papers)

S. James Gates, Jr., Yangrui Hu, Hazel Mak:

1. J. High Energ. Phys. 176 (2020). arXiv:1911.00807
2. J. High Energ. Phys. 89 (2020). arXiv:2002.08502
3. ATMP, 25(6), (2021), arXiv:2006.03609

Key Concerns:

1. 1 or 2 Adynkra diagrams for $\mathfrak{so}(n)$, $4 \leqslant n \leqslant 11$
2. Methodology:
   Branching rules for $\mathfrak{su}(n) \to \mathfrak{so}(n)$ via Young diagrams
3. Use of LieART program for Mathematica

Adynkra as a Hasse diagram

Natural grading: $\bigwedge^* S = \bigoplus_k \bigwedge^k S$

Level k of the diagram:

$$\boxed{\bigwedge^k S \;=\; \bigoplus_\lambda V(\lambda)}$$

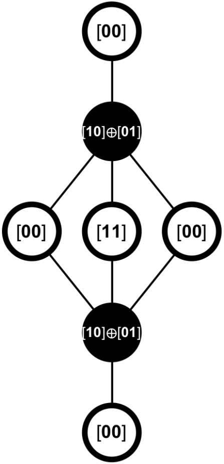
$V(\lambda) =$ irrep. of $\mathfrak{so}(n)$ with highest weight λ

We assign a **node** $V(\lambda)$ to each irreducible component of $\bigwedge^k S$, including multiplicities.

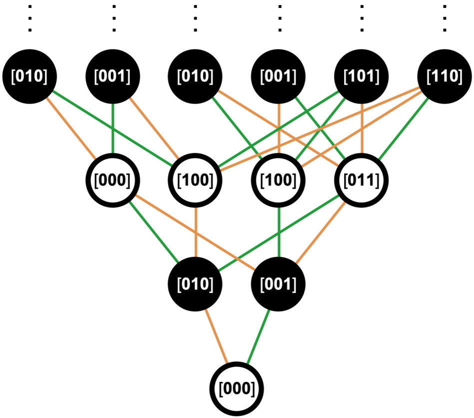
We **link** $\boxed{V(\lambda) \text{ in } \bigwedge^k S}$ to $\boxed{V(\mu) \text{ in } \bigwedge^{k+1} S}$

if $V(\mu)$ occurs in $\boxed{S \otimes V(\lambda) = \bigoplus_{\mu'} V(\mu')}$.

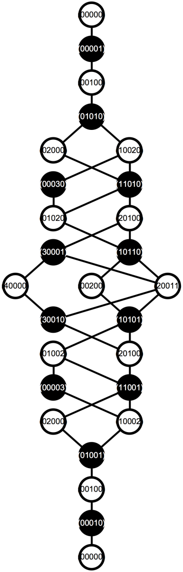
4D minimal scalar superfield (GHM, 2020-21)



6D minimal scalar superfield (GHM, 2020-21)



10D, $\mathcal{N} = 1$ Adynkra (GHM, 2020-21)



The Main Case: 11D, $\mathcal{N} = 1$

For the $11D$ case,

$$\dim\ S = 2^5 = 32$$

$$\dim\ \bigwedge^* S = 2^{32} = 4,294,967,296$$

Bosonic irreducible components: 1,494 unshaded nodes
Fermionic irreducible components: 1,186 shaded nodes

Middle level:

$$\dim\ \bigwedge^{16} S = C(32,16) = 601,080,390$$

Largest irreducible component has dimension 31,082,480

Components for $\bigwedge^{16} S$ (GHM, 2020-21)

Irreducible representations of $\mathfrak{so}(11)$ labeled by highest weight using fundamental weights (Dynkin labels)

(Multiplicity) Dykin label

- Level-16: (2)(00000) ⊕ (10000) ⊕ (20000) ⊕ (2)(00100) ⊕ (30000) ⊕ (2)(00010) ⊕ (00002) ⊕
(2)(40000) ⊕ (2)(02000) ⊕ (10100) ⊕ (50000) ⊕ (10010) ⊕ (3)(10002) ⊕ (2)(01100) ⊕ (60000) ⊕
(3)(20100) ⊕ (12000) ⊕ (31000) ⊕ (4)(00200) ⊕ (01010) ⊕ (4)(20010) ⊕ (70000) ⊕ (5)(01002) ⊕
(3)(20002) ⊕ (3)(00110) ⊕ (2)(00020) ⊕ (2)(30100) ⊕ (2)(00004) ⊕ (2)(22000) ⊕ (3)(11100) ⊕
(80000) ⊕ (3)(00102) ⊕ (00012) ⊕ (3)(10200) ⊕ (3)(30010) ⊕ (3)(02100) ⊕ (13000) ⊕ (5)(11010) ⊕
(2)(40100) ⊕ (4)(30002) ⊕ (32000) ⊕ (2)(04000) ⊕ (6)(11002) ⊕ (2)(21100) ⊕ (01200) ⊕ (5)(02010) ⊕
(5)(10110) ⊕ (2)(40010) ⊕ (2)(00300) ⊕ (3)(10020) ⊕ (50100) ⊕ (10004) ⊕ (42000) ⊕ (4)(20200) ⊕
(2)(40002) ⊕ (3)(02002) ⊕ (7)(10102) ⊕ (4)(21010) ⊕ (4)(10012) ⊕ (2)(12100) ⊕ (2)(31100) ⊕
(50010) ⊕ (6)(21002) ⊕ (3)(01110) ⊕ (60100) ⊕ (01020) ⊕ (03100) ⊕ (2)(00210) ⊕ (00006) ⊕
(00030) ⊕ (2)(50002) ⊕ (6)(20110) ⊕ (01004) ⊕ (2)(30200) ⊕ (6)(01102) ⊕ (4)(20020) ⊕ (2)(11200) ⊕
(4)(12010) ⊕ (00120) ⊕ (3)(31010) ⊕ (2)(00202) ⊕ (2)(20004) ⊕ (60010) ⊕ (10300) ⊕ (41100) ⊕
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(3)(20012) ⊕ (00112) ⊕ (2)(40200) ⊕ (2)(03002) ⊕ (00400) ⊕ (3)(30110) ⊕ (01300) ⊕ (30020) ⊕
(41010) ⊕ (4)(11110) ⊕ (21200) ⊕ (3)(22010) ⊕ (2)(11020) ⊕ (30004) ⊕ (10210) ⊕ (4)(30102) ⊕
(2)(41002) ⊕ (20300) ⊕ (6)(11102) ⊕ (2)(10030) ⊕ (2)(22002) ⊕ (02110) ⊕ (2)(30012) ⊕ (02020) ⊕
(3)(10202) ⊕ (2)(40110) ⊕ (3)(11012) ⊕ (40020) ⊕ (02102) ⊕ (02004) ⊕ (31200) ⊕ (2)(21110) ⊕
(40102) ⊕ (10112) ⊕ (20210) ⊕ (01202) ⊕ (3)(21102) ⊕ (20202) ⊕ (21012)

296 irreducible bosonic components in level 16

Three parts for validation

Project: Validate the Adynkra lists using alternative methods (dominant weight sets)

Three parts:

1. Dominant weight tables for irreducible representations (Mathematica/ LieART)
2. Character formulas for $\bigwedge^k S$ to obtain dominant weight tables (Maple)
3. Decompositions into irreducible components (Excel spreadsheet)

Weight Spaces for Representations

Example: $\mathfrak{so}(2N+1)$, type B_N

- Roots $\Delta = \{\pm e_i \pm e_j,\ \pm e_k \mid 1 \leqslant i, j, k \leqslant N,\ i \neq j, \}$,
- Positive roots $\Delta^+ = \{e_i \pm e_j,\ e_k \mid i < j\}$,
- Simple roots $\Pi = \{\alpha_i = e_i - e_{i+1}, \alpha_N = e_N\}$, and
- Weights: represent as N-tuple

$$a_1 e_1 + \cdots + a_N e_N \qquad \to \qquad (a_1, \ldots, a_N)$$

- Weight set for spin representation S:

$$\Gamma = \{\frac{1}{2}(\pm 1, \ldots, \pm 1)\}$$

with highest weight $\frac{1}{2}(1, \ldots, 1)$

Definitions

Many properties determined by Δ^+:

1) **Lexicographical ordering:** $\lambda_1 \leqslant \lambda_2$,

2) **Fundamental weights:** γ_i such that $\frac{2(\gamma_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{ij}$ for all simple α_j,

3) **Dominant weight:** sum of fundamental weights,

4) **Dynkin label:** describe sum of fundamental weights as N-tuple

$$a_1\gamma_1 + \ldots + a_N\gamma_N \quad \rightarrow \quad (a_1, \ldots, a_N) = (a_1 \ldots a_N)$$

Thus dominant weights have Dynkin labels with non-negative coefficients

Irreducibles and Highest Weights

If(π, V) is an **irreducible** representation of $\mathfrak{so}(2N+1)$,
then (π, V) is characterized by the set of dominant weights.

With respect to the lexicographical ordering,
there is a unique **highest weight** λ.
That is, $\lambda \geqslant \mu$ for all other weights μ.

It will be convenient to

1. denote an irreducible representations by its highest weight, and
2. describe this highest weight using a Dynkin label.

Example from LieART (Mathematica)

```
In[4]:= Dim[Irrep[B][2, 1, 0, 1, 2]]
Out[4]= 31082480

In[7]:= T4 = DominantWeightSystem[Irrep[B][2, 1, 0, 1, 2]]
```

2 1 0 1 2	1
0 2 0 1 2	1
2 1 0 2 0	1
0 2 0 2 0	1
1 0 1 1 2	2
2 1 1 0 2	2
0 2 1 0 2	2
1 0 1 2 0	2
2 1 1 1 0	3
3 0 0 0 4	3
0 2 1 1 0	3

Weight spaces for Exterior Products

(π, V) representation for $\mathfrak{so}(2N+1)$

Action of X in $\mathfrak{so}(2N+1)$ on $\bigwedge^k V$:

$$\begin{aligned} X(v_1 \wedge v_2 \wedge \cdots \wedge v_k) \quad = \quad & Xv_1 \wedge v_2 \wedge \cdots \wedge v_k \\ & + \; v_1 \wedge Xv_2 \wedge \cdots \wedge v_k \\ & + \; \ldots \\ & + \; v_1 \wedge v_2 \wedge \cdots \wedge Xv_k \end{aligned} .$$

If $\{\lambda_i\}$ is the weight set for V, including multiplicities, then the weights for $\bigwedge^k V$ are of the form

$$\lambda = \lambda_{i_1} + \cdots + \lambda_{i_k}$$

where the sum consists of k distinct λ_i

Character formula for $\bigwedge^k S$

Consider $t = (t_1, \ldots, t_N)$ as a N-tuple of variables.

Generating function for $\bigwedge^k S$

$$\boxed{\chi_k(t) = \sum_\lambda m_\lambda t^\lambda}$$

$m_\lambda =$ multiplicity of weight λ for $\bigwedge^k S$

Main character formula for $\bigwedge^* S$

$$\boxed{\sum_{k=0}^{d} z^k \chi_k(t) = \prod_{i=0}^{d} (1 + zt^{\lambda_i})}$$

Here $\chi_k(t) = e_k(t^{\lambda_1}, \ldots, t^{\lambda_d})$,
the k-th elementary symmetric function in d variables.

Set of interest: (λ, m_λ), the set of dominant weights with multiplicities

Algorithm for Tensor Product Decomposition

Let $\Omega(k)$ be the dominant weight set for $\bigwedge^k S$.

Algorithm:.
Partition $\Omega(k)$ into the dominant weight sets for irreducible $V(\lambda)$.

Recursive algorithm:

1. identify the largest weight λ,
2. extract the weight set for $V(\lambda)$ from $\Omega(k)$, and
3. repeat on the smaller set.

Example: 11D, $\bigwedge^2 S$

$$S = (00001), \qquad \dim S = 2^5 = 32$$

$$\bigwedge^2 S = (00010) \oplus (00100) \oplus (00000)$$

11D, Level 2							
dim as C(32,2)	496			dim	330	165	1
dim as sum	496			mult	1	1	1
					330	165	1
		Weight Multiplicities					
Dominant		Character	Irreps		Highest Weights in green		
Weight			Sum		00010	00100	00000
00000		16	16		10	5	1
10000		8	8		4	4	
01000		4	4		3	1	
00100		2	2		1	1	
00010		1	1		1		

Excel: $\Omega(2)$ for $\bigwedge^2 S$

Column 1: List of dominant weights from $\bigwedge^2 S$
Column 3: Weight multiplicities in purple cells

11D, Level 2							
dim as C(32,2)	496			dim	330	165	1
dim as sum	496			mult	1	1	1
					330	165	1
		Weight Multiplicities					
Dominant		Character	Irreps		Highest Weights in green		
Weight			Sum		00010	00100	00000
00000		16	16		10	5	1
10000		8	8		4	4	
01000		4	4		3	1	
00100		2	2		1	1	
00010		1	1		1		

Excel: Dominant weights for irreducibles

Column 1: List of dominant weights from $\bigwedge^2 S$

Green cells: Highest weights for irreducibles with
dominant weight multiplicities below

Column 4: sums from irreducibles, weighted by multiplicity (orange cells)

11D, Level 2							
dim as C(32,2)	496			dim	330	165	1
dim as sum	496			mult	1	1	1
					330	165	1
		Weight Multiplicities					
Dominant		Character	Irreps		Highest Weights in green		
Weight			Sum		00010	00100	00000
00000		16	16		10	5	1
10000		8	8		4	4	
01000		4	4		3	1	
00100		2	2		1	1	
00010		1	1		1		

Excel: Dimension check

$$C(32,2) = 496 \;=\; 330 \;+\; 165 \;+\; 1$$

Highest weights in green,
multiplicities of irreducibles in orange

11D, Level 2							
dim as C(32,2)	496			dim	330	165	1
dim as sum	496			mult	1	1	1
					330	165	1
		Weight Multiplicities					
Dominant		Character	Irreps		Highest Weights in green		
Weight			Sum		00010	00100	00000
00000		16	16		10	5	1
10000		8	8		4	4	
01000		4	4		3	1	
00100		2	2		1	1	
00010		1	1		1		

Thank you!