Counting formulas for anti-magic squares and conjoined compositions

Robert W. Donley, Jr.
Queensborough Community College (CUNY)

New York Combinatorics Seminar
CUNY Graduate Center, NYC

May 1, 2026

Table of Contents:

- Spring 2026: Queens Experience in Discrete Mathematics (QED) REU
- Spring 2025: Institute for Advanced Study / CUNY Fellowship Leave
- 2022/2024: Recruitment and Mentoring in Math. (RAMMP) REU

Girls' Angle Bulletin

http://girlsangle.org

Subscribe!

Search the Girls' Angle Bulletin:

ENHANCED BY G

Bulletin content and production is supported by generous grants from MathWorks.

Welcome

Membership

The Club

Mentors

Support Network

Girls' Angle Bulletin

The Girls' Angle Bulletin is a bimonthly math magazine with interviews, articles, problems, activities, art, and much more. Articles are written by professional mathematicians and scientists, our staff, students, and our members.

The Bulletin is a bridge. It bridges K12 students with mathematicians, and it bridges the gap between K12 math and the level of math expected of top college math majors. If you can read most of the Bulletin, you're ready to be a math major anywhere.

If you find content too hard, take a break - try again later. Girls' Angle members: Email us your questions. We will respond!

The electronic version is free and available on this page (click on a cover). For the print version, subscribe for $36/year or become a Bulletin Sponsor! The print version is ad-free, has higher quality graphics, and you'll be helping to further our mission.

Girls' Bulletin

The Latest Issue

Subscribe!

Check out this Girls' Angle blog post about the Bulletin.

Browse the Bulletin archive below or use the search box on the left.

Binomial Coefficient as Polynomial in t for fixed n

$$\binom{t+n}{n} = \frac{(t+n)!}{n!\ \ t!} = \frac{(t+n)(t+n-1)\dots(t+2)(t+1)}{n!}$$

When t is a non-negative integer:

- Number of ways to put t balls into $n+1$ boxes
- Number of **weak compositions** of t with $n+1$ parts
- (Number of solutions to $x_1 + x_2 + \cdots + x_{n+1} = L$ with $x_i \geqslant 0$)
- (Number of integer points in the n-simplex, dilated by a factor of L)

Weak composition: $(n+1)$-tuple of non-negative integers that sum to t

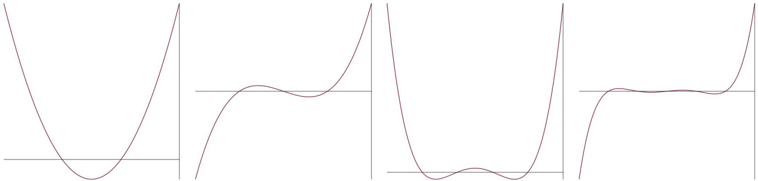
2	2	0	3	5

is a weak composition of 12 with 5 parts

Binomial Coefficient as Polynomial in t

$$Q_n(t) \quad = \quad \frac{(t+n)(t+n-1)\dots(t+2)(t+1)}{n!}$$

- $Q_n(0) = 1$
- $Q_n(t) = 0$ for $t = -1, -2, \dots, -n$
- Odd/evenness of graphs about the axis $x = -\frac{n+1}{2}$



Reciprocity for Binomial Coefficients

Replace t with $-t$:

$$\boxed{\binom{-t+n}{n} = (-1)^n \binom{t-1}{n}}$$

Proof:

$$\binom{-t+n}{n} = \frac{(-t+n)(-t+n-1)\ldots(-t+2)(-t+1)}{n!}$$

$$= (-1)^n \frac{(t-n)(t-n+1)\ldots(t-2)(t-1)}{n!}$$

$$= (-1)^n \binom{t-1}{n}. \qquad QED$$

Reciprocity for Binomial Coefficients

$$\boxed{\binom{-t+n}{n} \quad = \quad (-1)^n \quad \binom{(t-n-1)+n}{n}}$$

Interpret using combinatorial reciprocity:

Recall that the lefthand side is the number of weak compositions of t with $n+1$ parts.

Up to sign, the righthand side is the number of (strict) compositions of t with $n+1$ parts. In this case, only positive integers are allowed.

To see this, we first place a 1 in each of the $n+1$ boxes, reducing the available number of balls to $t-n-1$.

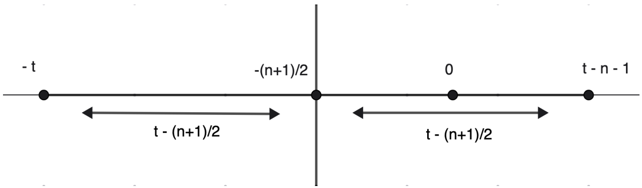
Reciprocity for Binomial Coefficients

$$\boxed{\binom{-t+n}{n} \quad = \quad (-1)^n \quad \binom{(t-n-1)+n}{n}}$$

$$\boxed{Q_n(-t) \quad = \quad (-1)^n \quad Q_n(t-n-1)}$$

Interpret as reflection:

Midpoint $(-t, \quad t-n-1) \quad = \quad -\frac{n+1}{2}$



Reciprocity for Binomial Coefficients

$$Q_n(t) \quad = \quad \frac{(t+n)(t+n-1)\dots(t+2)(t+1)}{n!}$$

Interpret as even/odd polynomial:

$Q_n(t)$ is even/odd as a function of $t + \frac{n+1}{2}$ or $2t + n + 1$

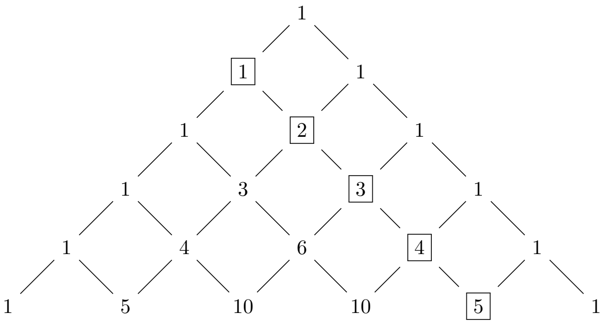
- $Q_0(t) = 1$
- $Q_1(t) = t + 1$
- $Q_2(t) = \frac{1}{2!}(t^2 + 3t + 2) \quad = \quad \frac{1}{8}[(2t + 3)^2 - 1]$
- $Q_3(t) = \frac{1}{3!}(t + 2)[(t + 2)^2 - 1]$
- $\dots$
- $Q_{2m-1}(t) = \frac{1}{(2m-1)!}(t + m)[(t + m)^2 - 1^2] \dots [(t + m)^2 - (m - 1)^2]$

Binomial Series and Pascal's Triangle

Binomial series:

$$\frac{1}{(1-x)^{n+1}} = 1 + \binom{n+1}{1}x + \binom{n+2}{2}x^2 + \cdots + \binom{n+k}{k}x^k + \dots$$

$n = 1:$



Binomial Series and Pascal's Triangle

Geometric series:

$$\boxed{\frac{1}{(1-x)} \quad = \quad 1 + x + x^2 + \dots}$$

Generating function for the sequence a_k:

$$F(x) \quad = \quad a_0 + a_1 x + a_2 x^2 + \dots$$

Then $\frac{1}{1-x} F(x)$ is the generating function for the sequence of partial sums

$$s_k = a_0 + a_1 + \dots + a_k$$

In turn, $\frac{1}{(1-x)^{n+1}} F(x)$ implements $n+1$ partial sum operations.

If $F(x) = 1$, we obtain the binomial series and hyperpyramidal figurate numbers (triangular numbers, tetrahedral numbers, etc.)

Binomial Series as Iterated Partial Sums

To the right: multiply by $\frac{1}{1-x}$

Partial Sum / Hockey Stick Rule:

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-4	-3	-2	-1	0	1	2	3	4
6	3	1	0	0	1	3	6	10
-4	-1	0	0	0	1	4	10	20
1	0	0	0	0	1	5	15	35
0	0	0	0	0	1	6	21	56

Pascal's Identity:

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-4	-3	-2	-1	0	1	2	3	4
6	3	1	0	0	1	3	6	10
-4	-1	0	0	0	1	4	10	20
1	0	0	0	0	1	5	15	35
0	0	0	0	0	1	6	21	56

Vandermonde Convolution

$$\boxed{\sum_{i+j=k} \binom{m}{i}\binom{n}{j} = \binom{m+n}{k}}$$

Proof 1: Disjoint union $X \cup Y$ with $|X| = m$, $|Y| = n$.

To choose k elements from $X \cup Y$,

first choose i from X, then $k - i$ from Y.

Then sum over i. QED

Proof 2: Match the coefficients of x^k in

$$(1+x)^m(1+x)^n = (1+x)^{m+n}.$$

QED

Vandermonde Convolution

1) VDC is a family of identities for fixed $m+n$ and k:
(m, n), $(m-1, n+1)$, $(m-2, n+2)$,...

$$(1+x)^m(1+x)^n \;=\; (1+x)^{m-1}(1+x)^{n+1} \;=\; \ldots \;=\; (1+x)^{m+n}$$

2) Bilinear form on power series: fix k

$$\langle p(x), q(x)\rangle_k \;=\; [x^k]p(x)q(x)$$

$$\langle p(x), q(x)\rangle_k \;=\; p_0q_k + p_1q_{k-1} + \cdots + p_kq_0$$

(Reverse one set of coefficients, then dot product)

Invariance under multiplication:

$$\langle h(x)p(x), q(x)\rangle_k = \langle p(x), h(x)q(x)\rangle_k$$

Special: $h(x) = \frac{1}{1-x}$ is the iterating factor for the table

Vandermonde Convolution (Partial Sums)

$$1 * 10 \;+\; 0 * 6 \;+\; 0 * 3 \;+\; 0 * 1 \;=\; 10$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-4	-3	-2	-1	0	1	2	3	4
6	3	1	0	0	1	3	6	10
-4	-1	0	0	0	1	4	10	20
1	0	0	0	0	1	5	15	35
0	0	0	0	0	1	6	21	56

$$1 * 4 \;+\; 1 * 3 \;+\; 1 * 2 \;+\; 1 * 1 \;=\; 10$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-4	-3	-2	-1	0	1	2	3	4
6	3	1	0	0	1	3	6	10
-4	-1	0	0	0	1	4	10	20
1	0	0	0	0	1	5	15	35
0	0	0	0	0	1	6	21	56

$$1 * 20 \;+\; (-1) * 10 \;+\; 0 * 4 \;+\; 0 * 1 \;=\; 10$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-4	-3	-2	-1	0	1	2	3	4
6	3	1	0	0	1	3	6	10
-4	-1	0	0	0	1	4	10	20
1	0	0	0	0	1	5	15	35
0	0	0	0	0	1	6	21	56

Semi-magic squares

A square matrix is called a **semi-magic square** if

1. entries are integers $\geqslant 0$, and
2. the sum along any row or column is equal to the same number L.

L is called the **line sum** of M.

Examples:

$$\begin{bmatrix} 3 & 1 & 2 \\ \boxed{2} & \boxed{2} & \boxed{2} \\ 1 & 3 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \boxed{0} & \boxed{0} & \boxed{1} & \boxed{1} \end{bmatrix}, \quad \begin{bmatrix} 1 & \boxed{2} & 3 & 4 \\ 2 & \boxed{3} & 4 & 1 \\ 3 & \boxed{4} & 1 & 2 \\ 4 & \boxed{1} & 2 & 3 \end{bmatrix}.$$

Some Highlights:

- P. McMahon (1916): Counting formula for size 3
- Coupling of angular momentum in particle physics (late 1920s)
- Birkhoff-Von Neumann Theorem (1946)
- Anand-Dumir-Gupta Conjecture (1966), extended and proved by R. Stanley
- M. Beck, D. Pixton (2003): Counting formula for size 9
- J. De Loera, F. Liu, R. Yoshida (2009): Generating function for general case

Example: Permutation Matrices

Size 3, $L = 1$

Rotations:

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Reflections:

$$P_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P_5 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Example: Sums of Permutation Matrices

Any linear combination with $a_i \geqslant 0$ integers

$$a_1 P_1 + \cdots + a_6 P_6$$

is a semi-magic square with line sum $L = a_1 + \cdots + a_6$.

Corollary of Birkhoff-Von Neumann Theorem

Every semi-magic square with line sum L is a sum of L permutation matrices.

Difficulty in general: linear dependence and relations over $\mathbb{N}$

Size 3: Unique relation

Solve:

$$\sum a_i P_i = \begin{bmatrix} a_1 + a_6 & a_3 + a_5 & a_2 + a_4 \\ a_2 + a_5 & a_1 + a_4 & a_3 + a_6 \\ a_3 + a_4 & a_2 + a_6 & a_1 + a_5 \end{bmatrix} = 0.$$

Solution:

$$a_1 = a_2 = a_3 = 1, \quad a_4 = a_5 = a_6 = -1,$$

or

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Consequences:

1) If some entry of M equals zero, then the sum is unique.

2) Otherwise, there are $m + 1$ distinct sums that represent M, where m is the smallest entry of M.

Rectangular notation

Every semi-magic square M can be represented by a rectangle:

$$M \quad \leftrightarrow \quad \mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6) \quad \leftrightarrow \quad \begin{array}{|c|c|c|} \hline a_1 & a_2 & a_3 \\ \hline a_4 & a_5 & a_6 \\ \hline \end{array}.$$

Here the single relation takes the form

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 1 & 1 & 1 \\ \hline \end{array}.$$

By repeatedly shifting up, uniquely represented if one of a_4, a_5, a_6 is zero.

Note: line sum $L = a_1 + \cdots + a_6$ is unchanged by shifting 1s

Anti-magic squares

Size 4 and beyond: Too many relations
(Anand-Dumir-Gupta Conjecture, extended and proved by R. Stanley)

Alternative model that generalizes the size 3 case:

Anti-magic squares (R. Stanley, EC1, Ch. 4, Problem 53)

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Example of Anti-magic Square

$$
\begin{aligned}
M &= (3C_1 + 4C_2 + 2C_3) + (0R_1 + 1R_2 + 2R_3) \\
&= (2C_1 + 3C_2 + 1C_3) + (1R_1 + 2R_2 + 3R_3) \\
&= (1C_1 + 2C_2 + 0C_3) + (2R_1 + 3R_2 + 4R_3)
\end{aligned}
$$

$$i(M) = 2 + 3 + 1 + 1 + 2 + 3 = 12$$

$$
\begin{bmatrix} \boxed{3} & 4 & 2 \\ 4 & \boxed{5} & 3 \\ 5 & 6 & \boxed{4} \end{bmatrix},
\begin{bmatrix} 3 & 4 & \boxed{2} \\ \boxed{4} & 5 & 3 \\ 5 & \boxed{6} & 4 \end{bmatrix},
\begin{bmatrix} 3 & \boxed{4} & 2 \\ 4 & 5 & \boxed{3} \\ \boxed{5} & 6 & 4 \end{bmatrix}
$$

$$
\begin{bmatrix} 3 & 4 & \boxed{2} \\ 4 & \boxed{5} & 3 \\ \boxed{5} & 6 & 4 \end{bmatrix},
\begin{bmatrix} 3 & \boxed{4} & 2 \\ \boxed{4} & 5 & 3 \\ 5 & 6 & \boxed{4} \end{bmatrix},
\begin{bmatrix} \boxed{3} & 4 & 2 \\ 4 & 5 & \boxed{3} \\ 5 & \boxed{6} & 4 \end{bmatrix}
$$

Rectangular notation

For size n, we obtain a $2 \times n$ rectangle by

$$M = c_1 C_1 + \cdots + c_n C_n + r_1 R_1 + \ldots r_n R_n$$

$$\to (c_1, \ldots, c_n, r_1, \ldots, r_n)$$

$$\to \begin{array}{|c|c|c|c|} \hline c_1 & c_2 & \ldots & c_n \\ \hline r_1 & r_2 & \ldots & r_n \\ \hline \end{array}$$

with the single relation

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & \ldots & 1 \\ \hline 0 & 0 & \ldots & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 0 & 0 & \ldots & 0 \\ \hline 1 & 1 & \ldots & 1 \\ \hline \end{array}.$$

Counting Anti-magic Squares (Exercise 53)

Upshifted rectangles. At least one zero in the bottom row.

Let $i(M) = \sum r_i + \sum c_i$ be the **index** of M,
and $\rho_n(L)$ the number of anti-magic squares of size n and index L.

$$\boxed{\rho_n(L) = \binom{L+2n-1}{2n-1} - \binom{L+n-1}{2n-1}}$$

Proof: The first term is the number of weak compositions (rectangles) of L with $2n$ parts.
The second term is the number such rectangles with non-vanishing parts in the bottom row.
(Pre-assign a ball to each box in the bottom row) QED

$$\boxed{F_n(x) = \frac{1-x^n}{(1-x)^{2n}} = \frac{1+x+\cdots+x^{n-1}}{(1-x)^{2n-1}}}$$

Conjoined Compositions

Extend to $m \times n$ rectangles:

Definition:

A **conjoined composition of type** (m, n) **and weight** L is a weak composition of L with mn parts subject to the shift relation.

Example: Type $(3, 4)$

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array}.$$

Explicit model

Generating set as *mn*-tuples

Type $(2,3)$:

$$110000, 001100, 000011$$

$$011000, 000110, 100001$$

Type $(3,2)$:

$$111000, 000111$$

$$110001, 001110$$

$$100011, 011100$$

General: rows of circulant semi-magic square

$$\begin{matrix} 1100 \\ 0110 \\ 0011 \\ 1001 \end{matrix}$$

Main Result:

Theorem: (QED REU 2026)

Let $\rho_{m,n}(L)$ be the number of conjoined compositions of type (m,n) with weight L. Then the generating function for $\rho_{m,n}(L)$ is given by

$$F_{m,n}(x) \quad = \quad \sum_{L=0}^{\infty} \rho_{m,n}(L) x^L \quad = \quad \frac{(1-x^n)^{m-1}}{(1-x)^{mn}}.$$

Anti-magic squares:

$$F_{2,n}(x) = \frac{1-x^n}{(1-x)^{2n}}.$$

Proof by induction

We count rectangles in upshifted form.

Base cases: Weak compositions / Anti-magic squares

Induction step: Assume the formula is true for CCs of type (m, n).

Type $(m+1, n)$: $F_{m+1,n}(x)$
Type (m, n): $F_{m,n}(x)$
Bottom row: a weak composition with n parts and at least one zero.

$$\rho(i) = \binom{i+n-1}{n-1} - \binom{i-1}{n-1} \qquad \rightarrow \qquad f(x) = \frac{1-x^n}{(1-x)^n}$$

Two independent choices: multiply the generating functions

$$\boxed{F_{m+1,n}(x) = \frac{1-x^n}{(1-x)^n} F_{m,n}(x)}$$

QED

Counting formulas: Anti-magic squares

$$\rho_n(L) = \binom{L+2n-1}{2n-1} - \binom{L+n-1}{2n-1}$$

Generating function:

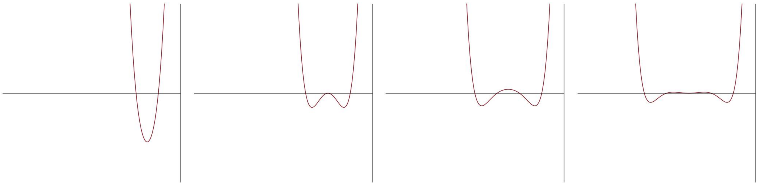
$$F_n(x) = \frac{1-x^n}{(1-x)^{2n}} = \frac{1+x+x^2+\cdots+x^{n-1}}{(1-x)^{2n-1}}$$

- $\rho_n(L)$ is a polynomial of degree $2n-2$
- $\rho_n(L) = 0$ for $L = -1, -2, \ldots, -(n-1)$
- $\rho_n(-L) = \rho(L-n)$

Counting formulas

$$\boxed{\rho_n(-L) = \rho_n(L-n)}$$

$\rho_n(L)$ is a even polynomial in $L + \frac{n}{2}$



Even as a Polynomial in $L + m$

$n = 2m$:

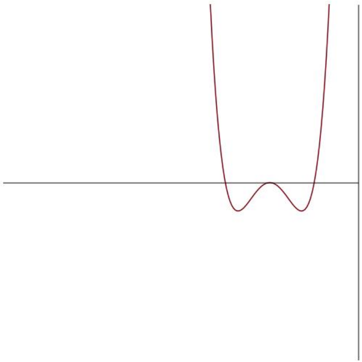
$$
\begin{aligned}
\rho_n(L) &= 2/(2n-1)! \\
&\times (L+m)^2[(L+m)^2 - 1^2]\dots[(L+m)^2 - (m-1)^2] \\
&\times [e_1(L+m)^{n-2} + e_3(L+m)^{n-4} + \cdots + e_{n-1}]
\end{aligned}
$$

where $e_k = e_k(m, m+1, \dots, 3m-1)$,
the k-th elementary symmetric function of degree $2m$

Even as a Polynomial in $L+2$

$$\rho_4(L) = 2/7! \;\; (L+2)^2 \; [(L+2)^2 - 1^2] \;\; [14(L+2)^2 + 154]$$

$$e_1(2,3,4,5) = 14, \quad e_3(2,3,4,5) = 154$$



Even as a Polynomial in $L+2$

Step 1:
Factor out

$$(L+1)(L+2)(L+3) = ((L+2)-1)(L+2)((L+2)+1)$$

Step 2:
Difference let $x = L+2$:

$$(L+7)(L+6)(L+5)(L+4) = (x+5)(x+4)(x+3)(x+2)$$

$$(L-3)(L-2)(L-1)(L) = (x-5)(x-4)(x-3)(x-2)$$

$$(x^4 + e_1x^3 + e_2x^2 + e_1x + e_0) - (x^4 - e_1x^3 + e_2x^2 - e_1x + e_0)$$
$$= 2e_1x^3 + 2e_3x$$

Generating function

$$F_{m,n}(x) \quad = \quad \frac{(1-x^n)^{m-1}}{(1-x)^{mn}} \quad = \quad \frac{(1+x+\cdots+x^{n-1})^{m-1}}{(1-x)^{mn-m+1}}$$

- $\rho_{m,n}(L)$ is a polynomial in L of degree $m(n-1)$
- $\rho_{m,n}(L) = 0$ for $L = -1, -2, \ldots, -(n-1)$
- $\rho_{m,n}(-L) \;=\; (-1)^{m(n-1)} \;\; \rho_{m,n}(L-n)$
- $F_{m,n}(x) \quad = \quad \frac{1-x^n}{(1-x)^n} \;\; F_{m-1,n}(x)$

Generating function: $n = 2$

$$\boxed{F_{m,2}(x) \quad = \quad \frac{(1-x^2)^{m-1}}{(1-x)^{2m}} \quad = \quad \frac{(1+x)^{m-1}}{(1-x)^{m+1}}}$$

- $\rho_{m,2}(L)$ is a polynomial in L of degree m
- $\rho_{m,2}(L) = 0$ for $L = -1$
- $\rho_{m,2}(-L) \;=\; (-1)^m\, \rho_{m,2}(L-2)$ (symmetric in L+1)
- $F_{m,2}(x) \;=\; \frac{1-x^2}{(1-x)^2}\; F_{m-1,2}(x) = \; \frac{1+x}{1-x}\; F_{m-1,2}(x)$

Generating function: $n = 2$

Explicit polynomials:

$$\boxed{\rho_{m,2}(L) = \binom{L+m}{m} + \binom{m-1}{1}\binom{L+m-1}{m} + \binom{m-1}{2}\binom{L+m-2}{m} + \dots}$$

$$m = 2 : \rho_{2,2}(L) = 1\binom{L+2}{2} + 1\binom{L+1}{2} = (L+1)^2$$

$$m = 3 : \rho_{3,2}(L) = 1\binom{L+3}{3} + 2\binom{L+2}{3} + 1\binom{L+1}{3} = \frac{1}{3}(L+1)[2(L+1)^2 + 1]$$

Factor:

$$\frac{1+x}{1-x} = 1 + 2x + 2x^2 + 2x^3 + \dots$$

Hockey stick rule:

$$s_k = a_k + 2a_{k-1} + 2a_{k-2} + 2a_{k-3} + \dots$$

Iterated Weighted Sums

To the right: multiply by $\frac{1+x}{1-x}$. Column zero is $\frac{1}{1-x^2}$

Hockey Stick Rule Variant: $44 = 16 + 2(9 + 4 + 1)$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-8	-6	-4	-2	0	2	4	6	8
33	19	9	3	1	3	9	19	33
-96	-44	-16	-4	0	4	16	44	96
225	85	25	5	1	5	25	85	225
-456	-146	-36	-6	0	6	36	146	456

Pascal's Identity Variant: $44 = 16 + 19 + 9$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-8	-6	-4	-2	0	2	4	6	8
33	19	9	3	1	3	9	19	33
-96	-44	-16	-4	0	4	16	44	96
225	85	25	5	1	5	25	85	225
-456	-146	-36	-6	0	6	36	146	456

Vandermonde Convolution

$$1*44 \;+\; 0*19 \;+\; 1*6 \;+\; 0*1 \;=\; 50$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-8	-6	-4	-2	0	2	4	6	8
33	19	9	3	1	3	9	19	33
-96	-44	-16	-4	0	4	16	44	96
225	85	25	5	1	5	25	85	225
-456	-146	-36	-6	0	6	36	146	456

$$1*16 \;+\; 2*9 \;+\; 3*4 \;+\; 4*1 \;=\; 50$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-8	-6	-4	-2	0	2	4	6	8
33	19	9	3	1	3	9	19	33
-96	-44	-16	-4	0	4	16	44	96
225	85	25	5	1	5	25	85	225
-456	-146	-36	-6	0	6	36	146	456

$$1*96 \;+\; (-2)*33 \;+\; 3*8 \;+\; (-4)*1 \;=\; 50$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-8	-6	-4	-2	0	2	4	6	8
33	19	9	3	1	3	9	19	33
-96	-44	-16	-4	0	4	16	44	96
225	85	25	5	1	5	25	85	225
-456	-146	-36	-6	0	6	36	146	456

Generating function: $n = 3$

$$F_{m,3}(x) = \frac{(1-x^3)^{m-1}}{(1-x)^{3m}} = \frac{(1+x+x^2)^{m-1}}{(1-x)^{2m+1}}$$

- $\rho_{m,3}(L)$ is a polynomial in L of degree $2m$
- $\rho_{m,3}(L) = 0$ for $L = -1, -2$
- $\rho_{m,3}(-L) = \rho_{m,3}(L-3)$ (even in L+3/2)
- $F_{m,3}(x) = \frac{1-x^3}{(1-x)^3} F_{m-1,3}(x) = \frac{1+x+x^2}{(1-x)^2} F_{m-1,3}(x)$

Generating function: $n = 3$

Explicit polynomials: (use unreduced form of $F_{m,3}(x)$)

$$\boxed{\rho_{m,3}(L) = \binom{L+3m-1}{3m-1} + \cdots + (-1)^k \binom{m-1}{k} \binom{L+3m-1-3k}{3m-1} + \ldots}$$

$$m = 3 : \rho_{3,3}(L) = 1\binom{L+8}{8} - 2\binom{L+5}{8} + 1\binom{L+2}{8}$$

$$= \frac{1}{8! \cdot 16}[(2L+3)^2 - 1][(2L+3)^6 + 421(2L+3)^4 + 25075(2L+3)^2 + 62055]$$

Factor:

$$\frac{1 + x + x^2}{(1-x)^2} = 1 + 3x + 6x^2 + 9x^3 + \ldots$$

Hockey stick rule:

$$s_k = a_k + 3a_{k-1} + 6a_{k-2} + 9a_{k-3} + \ldots$$

Iterated Weighted Sums

To the right: multiply by $\frac{1+x+x^2}{(1-x)^2}$. Column zero is $\frac{1}{1-x^3}$

Hockey Stick Rule Variant: $163 = 55 + 3(21) + 6(6) + 9(1)$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-12	-9	-6	-3	0	3	6	9	12
66	36	15	3	0	6	21	45	78
-215	-80	-17	1	1	10	55	163	361
435	90	-3	-6	0	15	120	477	1329
-462	18	39	6	0	21	231	1197	4134

Pascal's Identity Variant: $163 = 55 + 21 + 6 + 2(45) - 9$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-12	-9	-6	-3	0	3	6	9	12
66	36	15	3	0	6	21	45	78
-215	-80	-17	1	1	10	55	163	361
435	90	-3	-6	0	15	120	477	1329
-462	18	39	6	0	21	231	1197	4134

Vandermonde Convolution

$$1 * 163 \;+\; 0 * 45 \;+\; 0 * 9 \;+\; 1 * 1 \;=\; 164$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-12	-9	-6	-3	0	3	6	9	12
66	36	15	3	0	6	21	45	78
-215	-80	-17	1	1	10	55	163	361
435	90	-3	-6	0	15	120	477	1329
-462	18	39	6	0	21	231	1197	4134

$$1 * 55 \;+\; 3 * 21 \;+\; 6 * 6 \;+\; 10 * 1 \;=\; 164$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-12	-9	-6	-3	0	3	6	9	12
66	36	15	3	0	6	21	45	78
-215	-80	-17	1	1	10	55	163	361
435	90	-3	-6	0	15	120	477	1329
-462	18	39	6	0	21	231	1197	4134

$$1 * 361 \;+\; (-3) * 78 \;+\; 3 * 12 \;+\; 1 * 1 \;=\; 164$$

-4	-3	-2	-1	0	1	2	3	4
1	1	1	1	1	1	1	1	1
-12	-9	-6	-3	0	3	6	9	12
66	36	15	3	0	6	21	45	78
-215	-80	-17	1	1	10	55	163	361
435	90	-3	-6	0	15	120	477	1329
-462	18	39	6	0	21	231	1197	4134

Book References / Thank you!

- Ault, S., Kicey, C.: Counting Lattice Paths using Fourier Methods (2019)
- Barry P.: Riordan Arrays: A Primer (2016)
- Beck, M., S. Robins: Computing the Continuous Discretely. Integer point enumeration in polyhedra. (2007)
- Beck, M., R. Sanyal: Combinatorial Reciprocity Theorems. An invitation to enumerative geometric combinatorics. (2018)
- Stanley, R. P.: Enumerative Combinatorics, Volume 1. Second edition. Ch. 4 (2012)