INTRODUCTION TO GROUP REPRESENTATIONS
JUNE 11, 2012
LINEAR ALGEBRA REVIEW 1

Coordinates

Let V be a finite-dimensional vector space over $\mathbb{R}$, and suppose that $B = \{v_1, \ldots, v_n\}$ is a basis for V. Then we can write every v in V uniquely as

$$v = c_1 v_1 + \ldots c_n v_n$$

for some real c_i. We define a map to the coordinate space with respect to B

$$[\cdot]_B : V \to \mathbb{R}^n \quad \text{by} \quad v \mapsto [v]_B = (c_1, \ldots, c_n).$$

One sees immediately that $[\cdot]_B$ is a vector space isomorphism. If $V = \mathbb{R}^n$, then we note that

$$v = P_B[v]_B \quad \text{where} \quad P_B = [v_1 \ldots v_n].$$

One can regard this as a forerunner to the Fourier transform; in one direction, we strip out the coefficients with respect to a preferred basis, and, in the other, we reassemble the original vector, corresponding to Fourier inversion. More later on Fourier transforms.

Now suppose that $C = \{w_1, \ldots w_n\}$ is another basis for V. We have a map from B-coordinates to C-coordinates given by the map $P_{C \leftarrow B} = [\cdot]_C \circ [\cdot]_B^{-1}$, and one checks that

$$P_{D \leftarrow C} P_{C \leftarrow B} = P_{D \leftarrow B}.$$

If $V = \mathbb{R}^n$ then $P_{C \leftarrow B} = P_C^{-1} P_B$.

Associated Matrices

Now let $T : V \to V$ be a linear transformation. If we choose a basis B, then there is an associated linear transformation $[T]_B : \mathbb{R}^n \to \mathbb{R}^n$ on the coordinate space for B defined by

$$[T]_B([v]_B) := [Tv]_B = A_B[v]_B$$

where A_B is an $n \times n$ matrix with real entries. If $V = \mathbb{R}^n$ and B is the standard basis, then $A_B = [T(e_1) \ldots T(e_n)]$.

Suppose we change basis from B-coordinates to C-coordinates. Then

$$A_C[v]_C = [Tv]_C = P_{C \leftarrow B}[Tv]_B = P_{C \leftarrow B} A_B P_{C \leftarrow B}^{-1} [v]_C$$

or

$$A_C = P A_B P^{-1}$$

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gives the rule for changing the associated matrix when the basis changes. One notes that this is also the definition for similarity of square matrices:

$$A \sim B \quad \text{if and only if there exists an invertible } P \text{ such that } A = PBP^{-1}.$$

One shows that $\sim$ is a equivalence relation and the problem is to seek preferred candidates from each equivalence class.

Diagonal Form and Bases of Eigenvectors

A first preferred class is diagonal form. A diagonal matrix describes the simplest type of linear transformation: we merely rescale and/or reflect along each coordinate axis. In addition, many computations using diagonal matrices are straightforward. This suggests that we look for bases B that such that A_B is a diagonal matrix. Such a basis is called diagonalizing or an eigenbasis.

For now, suppose $V = \mathbb{R}^n$. If we wish to rescale or reflect along a direction, we are trying to solve the characteristic equation

$$Av = cv,$$

which is equivalent to the equation

$$(A - cI)v = 0.$$

If there is an nonzero solution, then we must have that $det(A - cI) = 0$. This leads us to define the characteristic polynomial $p_A(x) = det(xI - A)$. Note that we have switched the order to guarantee the leading term of the polynomial is 1. One should check that the characteristic polynomial is unaffected by change of basis and thus is defined for linear transformations in general. The roots of p_A are called the eigenvalues of A (or the characteristic roots of A) and these represent the scaling factors in the characteristic equation.

In turn, we find the rescaled directions by finding the null space $Null(A - cI)$ for each eigenvalue c. These are called the eigenvectors for A with eigenvalue c. If we can find a basis of eigenvectors, also called an eigenbasis, then the associated matrix is diagonal. We note that the diagonal entries are the eigenvalues in the same order as the basis of eigenvectors.

There are two drawbacks, illustrated by the following examples. First consider the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The characteristic polynomial is $x^2 + 1$, so there are no real roots and no eigenvalues over the reals; it can be diagonalized over the complex numbers. On the other hand, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has its eigenvalues in the reals, but there exists no eigenbasis. The eigenspace for eigenvalue 1 is one dimensional. Here we must go beyond diagonal form to Jordan canonical form or rational canonical form. Neither is needed for this course.

Invariant Subspaces

Invariant subspaces will be useful for the course. The eigenvector problem can be restated in terms of subspaces. Geometrically the characteristic equation $Av = cv$ (or $Tv = v$ for a linear transformation) says that the line through the origin in the v direction is mapped back to itself under A. That is, we say that $\mathbb{R}v$ is an invariant subspace under A. In general, we define a subspace W in V to be invariant under A if, whenever w in W, we have that Aw is also in W. Note that this is precisely the condition that allows us to restrict a linear transformation $T : V \to V$ to W :

$$T|_W : W \to W.$$

Invariance guarantees that the range is in W.

If we choose a basis $B' = \{w_1, \dots, w_k\}$ for W and complete it to a basis B for V, then for $w = b_1w_1 + \cdots + b_kw_k$ in W, $Aw = c_1w_1 + \cdots + c_kw_k + 0 + \cdots + 0$. Thus the associated matrix for A with respect to B is block upper-triangular

$$A_B = \begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix},$$

where A_1 is $k \times k$ and A_2 is $(n-k) \times (n-k)$. Note that we have no control over how the remaining basis elements map.

We say that V is a direct sum of two subspaces W and W' if every vector in V can be written uniquely as a sum $v = w + w'$ with w in W and w' in W'. Equivalently, if B is a basis for W and B' is a basis for W' then $B \cup B'$ is a basis for V. Equivalently, $W \cap W' = 0$ and $(dim\ W) + (dim\ W') = (dim\ V)$. Suppose this is the case and that also W and W' are both invariant under A. With respect to the basis $B \cup B'$ for V, the associated matrix for A is block diagonal:

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$