INTRO TO REP THEORY - JUNE 25, 2012
PROBLEM SET 3
RT3. EQUIVALENCE AND EXAMPLES

1. (a) Let π be the irreducible two-dimensional representation of Q, the quaternion group. Find orthonormal bases for $\pi(i)$ and $\pi(j)$. Recall that

$$\pi(i) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \pi(j) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

(b) Show that $(\pi, \mathbb{C}^2)$ is unitary and irreducible.

2. Consider the permutation representation of S_3 on $V = \{(x, y, z) \in \mathbb{C}^3 \mid x + y + z = 0\}$.

(a) Show that the vectors

$$v_1 = (\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}), \qquad v_2 = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$$

form an orthonormal basis B for V, and find $[\pi(23)]_B$ and $[\pi(123)]_B$.

(Hint: Use Fourier's Trick to find $[\pi(123)]_B$.)

(b) Find an equivalence between this representation and the representation σ of S_3 induced by the symmetries of an equilateral triangle on $\mathbb{R}^2$.

3. Let $G = \mathbb{Z}/4$ and let $V = \mathbb{C}^4$ with basis $B = \{e_0, e_1, e_2, e_3\}$. Let n in G act on V by extending the following action on basis vectors: $L(n)e_i = e_{i+n}$.

(a) Show that (L, V) is a faithful, unitary representation of G with respect to the inner product with B an orthonormal basis.

(b) Decompose (L, V) into a direct sum of irreducible representations of G.

(Hint: find an eigenvector basis with respect to $L(1)$.)

4. Repeat Problem 3 with $G = \mathbb{Z}/2,\ \mathbb{Z}/3$, and $\mathbb{Z}/2 \times \mathbb{Z}/2$.

5. Recall that the commutator subgroup $[G, G]$ of G is generated by all elements of the form $xyx^{-1}y^{-1}$ for x, y in G.

(a) Show that $[G, G] \triangleleft G$.

(b) Show that the quotient group $G/[G, G]$ is abelian. ("The abelianization of G")

Date: June 25, 2012.

(c) Show that if $N \triangleleft G$ and G/N is abelian, then $[G, G] \subseteq N$. Thus the abelianization of G is the largest abelian quotient of G.

(d) What happens if G is abelian? Simple?

6. (a) Compute the commutator subgroups of D_8, Q, A_4, A_5, D_{10}, and D_{12}.

(b) Find all one-dimensional representations for the groups in (a).

7. (a) Find orthogonal matrices representing the symmetries of a regular n-gon centered at the origin of $\mathbb{R}^2$. Orient with vertex 1 on the positive x-axis, increase labels counter-clockwise.

(b) Verify that (a) extends to a representation of D_{2n}, the dihedral group with $2n$ elements, on $\mathbb{C}^2$, and that this representation is unitary and irreducible.

8. If G is finite abelian, then FTFAG states that $G \cong \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_j$, where n_i divides n_{i+1}. Suppose the homomorphism $\pi_k : \mathbb{Z}/n \to \mathbb{C}^*$ is defined by $\pi_k(\omega) = \omega^k$ with $\omega = e^{2i\pi/n}$. Show that $G \cong G^*$, the character group of G, by sending

$$\mathbf{k} = (k_1, \ldots, k_j) \quad \text{to} \quad \pi_{\mathbf{k}}(i_1, \ldots, i_j) = \pi_{k_1}(i_1) \ldots \pi_{k_j}(i_j).$$

That is, show that

(a) G^* is a group under multiplication of functions,

(b) $\pi_{\mathbf{k}}$ is a character of G and

(c) the assignment is an isomorphism of groups, noting that $\pi_{\mathbf{k}}\pi_{\mathbf{k}'} = \pi_{\mathbf{k}+\mathbf{k}'}$.

(d) Verify that $G \cong G^*$ directly for $G = \mathbb{Z}/2,\ \mathbb{Z}/3,\ \mathbb{Z}/4$,and $\mathbb{Z}/2 \times \mathbb{Z}/2$.

9. Fix $n \geq 3$. Let $G = S_n$ act on $\mathbb{C}^n$ by permuting the labels of the standard basis.

(a) Show that this action π is a unitary representation on $\mathbb{C}^n$ with respect to the standard Hermitian inner product.

(b) Show that $sgn = det(\pi)$ is a representation of S_n.

(c) Let $W = \{(x_1, \ldots, x_n) \ : \ \sum x_i = 0\}$. Show that (π, W) is an irreducible subrepresentation.

(Hint: show that $(1, -1, \ldots, 0, 0)$ is in any nonzero subspace of W that is invariant under S_n.)

(d) Show that the trivial and sgn representation are the only one-dimensional representations of S_n.

10. Let (π, V) be a representation of G. Let σ be an element of $Aut(G)$.

(a) If σ is an inner automorphism, show that $\pi \circ \sigma$ is equivalent to π.

(b) The irreducible two-dimensional representation π of D_8 is equivalent to $\pi \circ \sigma$, where σ is the automorphism $r \mapsto r,\ c \mapsto cr$. Find an explicit equivalence.