INTRO TO GROUP REPS - JULY 9, 2012
PROBLEM SET 5
RT4.2. SCHUR'S LEMMA

1. Let (π, V) be a representation of G. Define the set of invariants in V under G, denoted V^G, to be the set of all v in V such that $\pi(g)v = v$ for all g in G.

(a) Show that (π, V^G) is a subrepresentation of G.

(b) If (π', W) is also a representation of G, show that $(Hom_{\mathbb{C}}(V, W))^G$ equals $Hom_G(V, W)$, the subspace of intertwining operators from V to W.

2. (a) Let D be a diagonal matrix with distinct diagonal entries. Show that if A is a matrix that commutes with D then A is also diagonal.

(b) Prove the converse of Schur's Lemma: if $Hom_G(V, V) = \mathbb{C}I$, then (π, V) is irreducible.

(c) Use (b) to show that the irreducible two-dimensional representations of S_3, D_8 and Q are irreducible using Schur's Lemma.

3. (a) Let $(\pi, \mathbb{C}^3)$ be the permutation representation of S_3. Compute $Hom_G(\mathbb{C}^3, \mathbb{C}^3)$. That is, find all matrices A that commute with all $\pi(g)$.

(b) Repeat (a) for the permutation representation of S_n on $\mathbb{C}^n$.

4. (a) Suppose (π, V) is irreducible. If z is in $Z(G)$, show that $\pi(z)$ is a multiple of the identity.

(b) Verify directly for Q, D_8, and D_{4n}.

5. Let (π, V) be a representation of G, and suppose S is any subset of V. We define the G-span of S, denoted $Span_G(S)$, as the smallest subspace containing $\pi(g)s$ for all s in S and g in G. If $Span_G(S) = V$, we say that S generates V.

(a) Show that $(\pi, Span_G(S))$ is a subrepresentation of (π, V).

(b) Show that π is irreducible if and only if $Span_G(v) = V$ for any nonzero v in V.

(c) For finite G, show that any irreducible representation is finite-dimensional with dimension $\leq |G|$.

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(d) Consider the permutation representation $(\pi, \mathbb{C}^3)$ on S_3. Find nonzero vectors v that generate and fail to generate $\mathbb{C}^3$.

6. If (π, V) is irreducible, show that the dimension of $Hom_G(V^m, V^n)$ is mn. Note that $V^m = V \oplus \cdots \oplus V$ (m times).

7. Let (π, V) be a unitary representation of G. An irreducible subrepresentation (π, V') is said to have type σ if (σ, W) is irreducible and $L : W \to V'$ is an equivalence. We denote V^σ the span of all subrepresentations of V of type σ.

(a) Show that $(\pi, (V^\sigma)^\perp)$ is a subrepresentation of (π, V) with no subrepresentation of type σ, and that $V = \oplus_\sigma V^\sigma$ where σ ranges over a set of inequivalent, irreducible representations of G.

(b) Suppose $V^\sigma = \oplus V_i$ and $V^\sigma = \oplus V'_i$ express V^σ as sums of irreducible subrepresentations of type σ. Show that there are an equal number of V_i and V'_i, both equal to the dimension of $Hom_G(W, V)$.

8. (a) Show that the map $c : GL(n, \mathbb{C}) \to GL(n, \mathbb{C})$ defined by $c(A) = (A^{-1})^T$ is a homomorphism. If $(\pi, \mathbb{C}^n)$ is a representation of G, show that $c \circ \pi$ is also.

(b) Suppose (π, V) is a representation of G, and let $B = \{u_i\}$ be a basis for V, and let $B^* = \{u_i^*\}$ be the corresponding dual basis. If (π^*, V^*) is the dual representation, show that

$$[\pi^*(g)]_{B^*} = [\pi(g^{-1}]_B^T.$$

(c) If π is unitary, use (b) to show that π^* is equivalent to $\overline{\pi}$, the complex conjugate representation of π.

(d) Verify (c) for one-dimensional representations, and use (c) to show why the defining representation of D_{2n} is equivalent to its dual.

9. Suppose (π, V) is unitary representation with invariant Hermitian inner product $\langle \cdot, \cdot \rangle$. Let $Sesq(V)$ denote the vector space of sesquilinear forms on V; that is, all functions f from $V \times V$ to $\mathbb{C}$ such that

(1) $f(v + v', w) = f(v, w) + f(v', w)$,
(2) $f(v, w + w') = f(v, w) + f(v, w')$, and
(3) $f(cv, w) = cf(v, w) = f(v, \overline{c}w)$.

(a) Show that $[\sigma(g)f](v, w) = f(\pi(g)^{-1}v, \pi(g)^{-1}w)$ defines a representation of G on $Sesq(V)$.

(b) Show that $i : Hom_{\mathbb{C}}(V, V) \to Sesq(V)$ defined by $i(T)(v, w) = \langle Tv, w \rangle$ is an equivalence of representations. How should we define an inner product on $Sesq(V)$ to obtain a unitary equivalence?

(c) If X is a Hermitian linear transformation in (b), what property does $i(X)$ possess?

(d) Identify the set of invariants (Problem 1) in $Sesq(V)$. What if π is irreducible?

10. Suppose (π, V) and (π', V') are inequivalent, irreducible, unitary representations of G. Let $\{u_i\}$ be an orthonormal basis for V', and let $\langle \cdot, \cdot \rangle$ be any sesquilinear pairing between V and V'.

(a) Define $T : V \to V'$ by

$$Tv = \sum_i \langle v, u_i \rangle u_i.$$

Show that the definition of T is independent of the orthonormal basis chosen.

(b) If we choose othonormal bases B and C for V and V', describe the associated matrix $[T]_{B,C}$.

(c) Show that only invariant sesquilinear pairing between V and V' is 0. See Problem 9 for definition, only now functions are on $V \times V'$.