# INTRO TO REP THEORY - JUNE 25, 2012
# SOLUTION SET 3
# RT. EQUIVALENCE AND EXAMPLES

1. (a) Recall that

$$\pi(i) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \pi(j) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

$\pi(j)$ is diagonal, so the standard basis $\{e_1, e_2\}$ works. Noting that $\pi(i)$ is rotation about the origin by $\pi/2$ clockwise, we may use the unit eigenvectors $(1/\sqrt{2}, i/\sqrt{2})$ and $(1\sqrt{2}, -i/\sqrt{2})$ from Solution Set 1, Problem 2(a).

(b) Since $Q$ is generated by $i$ and $j$, it is enough to show $\pi(i)$ and $\pi(j)$ are unitary matrices. Both satisfy $\pi(g)\pi(g)^* = I$, so $\pi$ is unitary. Since $\pi(j)$ is diagonal with distinct eigenvalues, $\pi(i)$ shares no eigenspaces with $\pi(j)$, and there are no one-dimensional subrepresentations. Thus $\pi$ is irreducible.

2. (a) We have $B = \{v_1, v_2\}$ where

$$v_1 = (\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}), \qquad v_2 = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}).$$

First $[\pi(23)]_B = diag(1, -1)$. Using Fourier's Trick, we have

$$\pi(123)v_1 = -\frac{1}{2}v_1 + \frac{\sqrt{3}}{2}v_2, \qquad \pi(123)v_2 = -\frac{\sqrt{3}}{2}v_1 + \frac{1}{2}v_2.$$

Thus $[\pi(123)]_B = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$, the rotation matrix for $\frac{2\pi}{3}$ counter-clockwise.

(b) All the work is done in part (a). Explicitly, define $L : V \to \mathbb{C}^2$ by $L(v_1) = e_1$ and $L(v_2) = e_2$. On $\mathbb{C}^2$, the group actions match: $\sigma(g) = [\pi(g)]_B$.

3. (a) Note that, with respect to $B$, $\pi(1) = A$ from Solution Set 1, Problem 3. Since $A^4 = I$ with smallest positive exponent, $\pi$ is faithful, and, since $AA^* = I$, $\pi$ is unitary.

(b) Referring to Solution Set 1, Problem 3 again, the span of a given eigenvector for $A$ corresponds to a one-dimensional subrepresentation of $\mathbb{Z}/4$. Checking Solution Set 1, Problem 8(b), we see that $\chi(1)$ corresponds to the eigenvalue, so each one-dimensional representation occurs exactly once.

4. As with Problem 3, check Solution Set 1, Problem 8(b) for eigenvectors and eigenvalues.

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5. (a) Since $[G, G]$ is generated by the elements $xyx^{-1}y^{-1}$, it is enough to show that these generators are preserved by conjugation. Thus

$$g(xyx^{-1}y^{-1})g^{-1} = (gxg^{-1})(gyg^{-1})(gxg^{-1})^{-1}(gyg^{-1})^{-1} \in [G, G].$$

(b) Let $N = [G, G]$. Since $N$ is a subgroup, we have $N = gN$ for any $g$ in $N$. Now

$$(xN)(yN) = xyN = xy(y^{-1}x^{-1}yxN) = yxN = (yN)(xN).$$

(c) We have $(xN)(yN) = (yN)(xN)$, so $xyN = yxN$, or $x^{-1}y^{-1}xyN = N$. Thus $x^{-1}y^{-1}xy$ is in $N$ for all $x, y$, so $[G, G] \subseteq N$.

(d) If $G$ is abelian, $xyx^{-1}y^{-1} = e$ and $[G, G]$ is trivial. So $G$ is isomorphic to is its own abelianization. If $G$ is simple and non-abelian, then $G = [G, G]$ by part (a). Thus the abelianization is the trivial group.

6. (a) We have

(1) $D_8 : \{e, r^2\}$,
(2) $Q : \{\pm 1\}$,
(3) $A_4$: $\{e, (12)(34), (13)(24), (14)(23)\}$,
(4) $A_5 : A_5$ (simple, non-abelian),
(5) $D_{10} : \langle r \rangle$, and
(6) $D_{12} : \langle r^2 \rangle$.

(b) These arise from composing $q : G \to G/[G, G]$ with characters of $G/[G, G]$. The abelianizations for each case are:

(1) $D_8 : \mathbb{Z}/2 \times \mathbb{Z}/2$, (note SS1, Problem 8(b) and SS1, Problem 10(b)),
(2) $Q : \mathbb{Z}/2 \times \mathbb{Z}/2$, (note SS1, Problem 8(b) and SS1, Problem 10(c)),
(3) $A_4$: $\mathbb{Z}/3$, (note SS1, Problem 8(b)),
(4) $A_5$ : trivial only,
(5) $D_{10} : \mathbb{Z}/2$, and
(6) $D_{12} : \mathbb{Z}/2 \times \mathbb{Z}/2$.

7. (a) Let $c$ represent the reflection across the $x$-axis and $r$ the rotation by $\theta = \frac{2\pi}{n}$ counterclockwise about the origin. With respect to the standard basis,

$$\pi(c) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \pi(r) = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}.$$

Now every element of $D_{2n}$ is of the form $c^i r^j$ with $i$ in $\{0, 1\}$ and $j$ in $\{0, \ldots, n-1\}$. We note that half the elements are rotations and half are reflections.

(b) Since $D_{2n}$ is generated by $c$ and $r$, $\pi$ is unitary since $\pi(c)$ and $\pi(r)$ are. As before, we have irreducibility since $\pi(c)$ is diagonal with distinct eigenvalues, but shares no eigenspaces with $\pi(r)$. So no one-dimensional subrepresentations.

8. (a) We show the group properties for $G^*$:

(1) Closure: we show that $\chi_1\chi_2$ is a character. If $g, h$ in $G$, then

$$(\chi_1\chi_2)(gh) = \chi_1(gh)\chi_2(gh) = (\chi_1\chi_2)(g)(\chi_1\chi_2)(h),$$

(2) Associative: property of multiplication of functions,
(3) Identity: the trivial character $\chi_0(g) = 1$ for all $g$ in $G$, and
(4) Inverses: because $|\chi(g)| = 1$ for all $g$ in $G$, $\chi^{-1} = 1/\chi = \overline{\chi}$.

(b) we must check the homomorphism property for $\pi_{\mathbf{k}}$, which is straightforward,

(c) again the homomorphism property holds: $\mathbf{k} + \mathbf{k}' \mapsto \pi_{\mathbf{k}+\mathbf{k}'}$. Since both sides have equal order, it is enough to show the assignment is one-one. If $\pi_{\mathbf{k}}(g) = 1$ for all $g$ in $G$, then each $\pi_{k_i} = 1$ by restricting to $\mathbb{Z}/n_i$. Thus $k_i = 0$ for all $i$.

(d) See SS1, Problem 8(b). The result is clear for $\mathbb{Z}/2$ and $\mathbb{Z}/3$. For $\mathbb{Z}/4$, $(\chi_1)^i = \chi_i$. For $\mathbb{Z}/2 \times \mathbb{Z}/2$, each $\chi_i^2 = \chi_0$.

9. (a) If we permute coordinates of $v$ and $w$ in a similar manner, the sum in $\langle v, w\rangle$ is unchanged. So $\pi$ is unitary.

(b) Since $det(AB) = det(A)det(B)$, $sgn$ is a homomorphism as a composition of homomorphisms.

(c) If we show that $v = (1, -1, 0, \ldots, 0)$ is in some $V$ invariant under $\pi$, then permuting shows each $v_i = \pi[(1i)(2\ i{+}1)]v = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$ is in $V$. But these $n-1$ vectors form a basis of $W$.

Suppose $w$ is nonzero in $V$. We proceed by induction on the number of nonzero coordinates. If $n = 2$, then, after permuting, $w = (x, -x, 0, \ldots, 0)$ for some nonzero $x$, and rescaling shows $v$ is in $V$.

Suppose the result is true if there are $\leq k$ nonzero coordinates, and that $w$ has $k+1$ nonzero coordinates. Permuting and rescaling, we may assume $w = (1, x_1, \ldots, x_k, 0, \ldots, 0)$ with $x_1 < 0$. If $x_1 = -1$, then $w + \pi(12)w$ has exactly $k-1$ nonzero coordinates, and the induction hypothesis applies. Otherwise $w' = -x_1 w + \pi(12)w$ has less than or equal to $k-1$ nonzero coordinates and is in $W$. Note that $w'$ is not exactly zero; otherwise, each nonzero $x_i$ satisfies $-x_1x_i + x_i = 0$. Again the induction hypothesis applies, and the result follows.

(d) Since $[S_n, S_n] = A_n$, the abelianization is isomorphic to $\mathbb{Z}/2$. Thus there are only two distinct characters.

10. (a) Suppose that $\sigma(g) = xgx^{-1}$ for some $x$ in $G$. Consider $L : V \to V$ defined by $L(v) = \pi(x)v$. We show that $L$ is an equivalence; that is, $L$ is a vector space isomorphism and an intertwining operator between $\pi$ and $\pi \circ \sigma$. First $L$ is an isomorphism since $\pi(x)$ is invertible. Now

$$(\pi \circ \sigma)(g)L(v) = \pi(xgx^{-1})\pi(x)v = \pi(x)\pi(g)v = L(\pi(g)v)$$

for any $g$ in $G$, so $L$ is an intertwining operator.

(b) Sketching axes of reflection, we see that this automorphism is induced by rotating the plane by $\frac{\pi}{4}$ clockwise. So let $L : \mathbb{C}^2 \to \mathbb{C}^2$ be given by

$$Lv = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} v.$$

$L$ is an isomorphism since $det(L) = 1$, and the intertwining property follows by showing

$$(\pi \circ \sigma)(r)L = \pi(r)L = L\pi(r) \quad \text{and} \quad (\pi \circ \sigma)(c)L = \pi(cr)L = L\pi(c).$$