INTRO TO GROUP REPS - AUGUST 20, 2012
SOLUTION SET 11
RT9. BASIC TENSOR ANALYSIS

1. (a) If some $a_i = a_j$, then the product equals zero. On the other hand, V has two identical rows, which implies $det(V) = 0$.

(b) Proof by induction on n. If $n = 2$, then $det(V) = a_2 - a_1$.

Suppose the statement is true whenever V is $n \times n$. Suppose V is $(n+1) \times (n+1)$. Using the alternating sum definition of determinant, the degree of $p(x)$ is at most n. Now $p(x)$ has zeros $a_1, \ldots, a_n$, so

$$p(x) = c_n(x - a_1) \ldots (x - a_n).$$

Set $x = 0$. Then $p(0) = (-1)^n c_n a_1 \ldots a_n$. On the other hand, taking the determinant of the corresponding matrix gives $(-1)^n a_1 \ldots a_n det(V_n)$.

If some $a_k = 0$, the induction step holds. Otherwise the induction hypothesis implies

$$c_n = det(V_n) = \prod_{i<j}(a_j - a_i).$$

Setting $x = a_{n+1}$ proves the induction step in this case.

2. (a) By Problem 1,

$$det(V(1, \omega, \ldots, \omega^{n-1})) = \prod_{i<j}(\omega^j - \omega^i).$$

By SS9, Problem 2(b), $|det(V)|^2 = n^n$, and the result follows.

(b) $n = 2$: Roots are ± 1. $2^1 = 1 - (-1)$.

$n = 3$: Roots are $1,\ \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ \omega^2 = \overline{\omega}$.

$$|(\omega^2 - \omega)(\omega^2 - 1)(\omega - 1)| = |(\omega^2 - 1)(\omega - 1)^2| = 3^{3/2}.$$

$n = 4$: Roots are $\pm 1, \pm i$.

$$|(-i + 1)(-i - i)(-i - 1)(-1 - i)(-1 - 1)(i - 1)| = 16 = 4^2.$$

$n = 6$: Roots are $1,\ \omega = \frac{1}{2} + \frac{\sqrt{3}}{2}i,\ \omega^2, \omega^3 = -1,\ \omega^4,\ \omega^5$. Inside the modulus, we can factor powers of ω out of each term. In steps,

$$|(\omega^5 - \omega^4)(\omega^5 - \omega^3)(\omega^5 - \omega^2)(\omega^5 - \omega)(\omega^5 - 1)| = 6,$$

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$$|(\omega^4-\omega^3)(\omega^4-\omega^2)(\omega^4-\omega)(\omega^4-1)|=6,$$

$$|(\omega^3-\omega^2)(\omega^3-\omega)(\omega^3-1)|=2\sqrt{3},$$

$$|(\omega^2-\omega)(\omega^2-1)(\omega-1)|=\sqrt{3}.$$

Taking the product gives

$$6\cdot 6\cdot 2\sqrt{3}\cdot\sqrt{3}=6^3.$$

3. (a) Let $N=|G|$. Consider the equation $\pi(g)^N=I$. Choose an orthonormal basis for V and consider the equation determined by the $(1,1)$-th entry. For all g in G,

$$\sum \phi_{*,*}(g)\dots\phi_{*,*}(g)=1$$

where each terms in the sum is a product of $|G|$ matrix coefficients of π. Using the identity

$$\phi_{u,v}(g)\phi_{u',v'}(g)=\langle\pi(g)u,v\rangle\langle\pi(g)u',v'\rangle=\langle(\pi\otimes\pi)(g)u\otimes u',v\otimes v'\rangle,$$

the left-hand side of the equation is a sum of matrix coefficients for $\otimes^N\pi$. Thus the constant functions are in the span of matrix coefficients for $\otimes^N\pi$ and this tensor product contains the trivial representation of G.

If π is irreducible and $N=2$, then Schur's Lemma implies that $V^*\cong V$. That is, the character of π is real-valued.

Checking characters in SS9, Problem 10, $\pi_1\otimes\pi_1$ does not contain the trivial representation, but $\otimes^3\pi_1$ does with multiplicity 2.

(b) If $\otimes^N\pi$ contains the trivial representation as in part (a), then Schur's Lemma implies that some subrepresentation of $\otimes^{N-1}\pi$ is equivalent to π^*.

(c) A faithful character of $\mathbb{Z}/n$ is of the form $\chi_k(1)=(e^{2\pi i/n})^k$ where $(k,n)=1$. Since k is a generator for $\mathbb{Z}/n$ by Bezout's Identity, the result follows.

4. Recall the character table for A_4 :

A_4	e (1)	(123) (4)	(132) (4)	(12)(34) (3)
χ_0	1	1	1	1
χ_1	1	ω	ω^2	1
χ_2	1	ω^2	ω	1
π	3	0	0	-1

We apply the formulas

$$\chi_{\wedge^2\pi}(g)=\frac{1}{2}[(\chi_\pi(g))^2-\chi_\pi(g^2)],\qquad \chi_{\odot^2\pi}(g)=\frac{1}{2}[(\chi_\pi(g))^2+\chi_\pi(g^2)].$$

A_4	e (1)	(123) (4)	(132) (4)	(12)(34) (3)
π	3	0	0	-1
$\pi \otimes \pi$	9	0	0	1
$\chi_\pi(g^2)$	3	0	0	3
$\wedge^2\pi$	3	0	0	-1
$\odot^2\pi$	6	0	0	2

Thus $\wedge^2\pi \cong \pi$. The sum of the squares of multiplicities for $\odot^2\pi$ is 4. Since the trivial type occurs exactly once, each irreducible type occurs with multiplicity one. That is

$$\odot^2\pi \cong \chi_0 \oplus \chi_1 \oplus \chi_2 \oplus \pi.$$

5. Recall the character table for S_4:

S_4	e (1)	(12) (6)	(123) (8)	(1234) (6)	(12)(34) (3)
χ_{triv}	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
π_1	3	1	0	-1	-1
π_2	2	0	-1	0	2
$\chi_{sgn} \otimes \pi_1$	3	-1	0	1	-1

First note that

$$\wedge^2\chi_{triv} \cong \wedge^2\chi_{sgn} = 0 \qquad \text{and} \qquad \odot^2\chi_{triv} \cong \odot^2\chi_{sgn} \cong \chi_{triv}.$$

S_4	e (1)	(12) (6)	(123) (8)	(1234) (6)	(12)(34) (3)
π_1	3	1	0	-1	-1
$\pi_1 \otimes \pi_1$	9	1	0	1	1
$\chi_{\pi_1}(g^2)$	3	3	0	-1	3
$\wedge^2\pi_1$	3	-1	0	1	-1
$\odot^2\pi_1$	6	2	0	0	2

We see immediately that $\wedge^2\pi_1$ is equivalent to $sgn \otimes \pi_1$. Checking character length squared, the sum of squares of multiplicities is 3, so there are 3 distinct types with multiplicity one. More checking shows

$$\odot^2\pi_1 \cong \chi_{triv} \oplus \pi_1 \oplus \pi_2.$$

This result also applies to $\chi_{sgn} \otimes \pi_1$. Next

S_4	e (1)	(12) (6)	(123) (8)	(1234) (6)	(12)(34) (3)
π_2	2	0	-1	0	2
$\pi_2 \otimes \pi_2$	4	0	1	0	4
$\chi_{\pi_2}(g^2)$	2	2	-1	2	2
$\wedge^2\pi_2$	1	-1	1	-1	1
$\odot^2\pi_2$	3	1	0	1	3

One sees immediately that $\wedge^2\pi_2 \cong sgn$ and $\odot^2\pi_2 \cong \chi_{triv} \oplus \pi_2$.

6. (a) The commutator subgroup of S_5 is A_5, and the abelianization of S_5 is isomorphic to $\mathbb{Z}/2$.

S_5	e (1)	(12) (10)	(123) (20)	(1234) (30)	(12345) (24)	(12)(34) (15)	(123)(45) (20)
χ_{triv}	1	1	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1	1	-1
π	5	3	2	1	0	1	0
π_4	4	2	1	0	-1	0	-1

(b)

S_5	e (1)	(12) (10)	(123) (20)	(1234) (30)	(12345) (24)	(12)(34) (15)	(123)(45) (20)
π_4	4	2	1	0	-1	0	-1
$\pi_4 \otimes \pi_4$	16	4	1	0	1	0	1
$\pi_4(g^2)$	4	4	1	0	-1	4	1
$\odot^2 \pi_4$	10	4	1	0	0	2	1

The length squared of the last row is $100 + 10(16) + 20(1) + 15(4) + 20(1) = 360$. Thus there are three irreducible types with multiplicity one each. One checks that two of these are equivalent to the trivial type and π_4. Subtracting gives

S_5	e (1)	(12) (10)	(123) (20)	(1234) (30)	(12345) (24)	(12)(34) (15)	(123)(45) (20)
χ_{triv}	1	1	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1	1	-1
π_4	4	2	1	0	-1	0	-1
π_5	5	1	-1	-1	0	1	1

(c)

S_5	e (1)	(12) (10)	(123) (20)	(1234) (30)	(12345) (24)	(12)(34) (15)	(123)(45) (20)
χ_{triv}	1	1	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1	1	-1
π_4	4	2	1	0	-1	0	-1
π_5	5	1	-1	-1	0	1	1
$s \otimes \pi_4$	4	-2	1	0	-1	0	1
$s \otimes \pi_5$	5	-1	-1	1	0	1	-1

(d)

S_5	e (1)	(12) (10)	(123) (20)	(1234) (30)	(12345) (24)	(12)(34) (15)	(123)(45) (20)
π_4	4	2	1	0	-1	0	-1
$\pi_4 \otimes \pi_4$	16	4	1	0	1	0	1
$\pi_4(g^2)$	4	4	1	0	-1	4	1
$\wedge^2 \pi_4$	6	0	0	0	1	-2	0

The length squared of the last row is 1, so $\wedge^2\pi_4$ is irreducible. Summarizing, we have

S_5	e (1)	(12) (10)	(123) (20)	(1234) (30)	(12345) (24)	(12)(34) (15)	(123)(45) (20)
χ_{triv}	1	1	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1	1	-1
π_4	4	2	1	0	-1	0	-1
π_5	5	1	-1	-1	0	1	1
$s \otimes \pi_4$	4	-2	1	0	-1	0	1
$s \otimes \pi_5$	5	-1	-1	1	0	1	-1
$\wedge^2\pi_4$	6	0	0	0	1	-2	0

7. (a) Since sgn equals one on A_5, we ignore upon restriction to A_5.

A_5	e (1)	(123) (20)	(12345) (12)	(12354) (12)	(12)(34) (15)
χ_{triv}	1	1	1	1	1
π_4	4	1	-1	-1	0
π_5	5	-1	0	0	1
$\wedge^2\pi_4$	6	0	1	1	-2

The length squared of the last row is 120, so $\wedge^2\pi_4$ has two inequivalent irreducible types. Since neither is trivial, both must have dimension 3. If we consider the lengths squared of columns, the characters on (123) equal zero. Consideration of inner products with the character of π_5 imply the value of the characters on (12)(34) equals -1. Since the classes for 5-cycles are closed under inverses, these character values are real.

Character length implies

$$9 + 12|\tau|^2 + 12|\tau'|^2 + 15 = 60 \quad \text{or} \quad \tau^2 + \tau'^2 = 3.$$

Inner product with the trivial character implies

$$3 + 12\tau + 12\tau' - 15 = 0 \quad \text{or} \quad \tau + \tau' = 1.$$

Substituting we have

$$\tau^2 + (1-\tau)^2 = 3 \quad \text{or} \quad 2\tau^2 - 2\tau - 2 = 0 \quad \text{or} \quad \tau = \frac{1 \pm \sqrt{5}}{2}.$$

Note that the equations are satisfied by allowing τ and τ' to be either root.

A_5	e (1)	(123) (20)	(12345) (12)	(12354) (12)	(12)(34) (15)
χ_{triv}	1	1	1	1	1
π_4	4	1	-1	-1	0
π_5	5	-1	0	0	1
π_3	3	0	τ	τ'	-1
π_3'	3	0	τ'	τ	-1

(b) We compute the character by considering eigenvalues for a representative rigid motion. For (123), we can use any rotation about a triangle; the eigenvalues are 1 and $\frac{-1\pm\sqrt{3}i}{2}$.

Thus the trace equals zero. For (12)(34), the rigid motion is a rotation by π; the eigenvalues are 1, -1, -1, so the trace equals -1. Suppose (12345) corresponds to a rotation about a vertex by $2\pi/5$. Then the eigenvalues are $1, e^{\pm 2\pi i/5}$, and the trace is $1+2cos(2\pi/5)$. With this choice, (12354) corresponds to rotation by $4\pi/5$, and the trace is $1+2cos(4\pi/5)$.

A_5	e (1)	(123) (20)	(12345) (12)	(12354) (12)	(12)(34) (15)
σ	3	0	$1+2cos(2\pi/5)$	$1+2cos(4\pi/5)$	-1

Since this character is not a multiple of the trivial character, it must correspond to an irreducible representation. Note that we could choose (12345) to represent a rotation by $4\pi/5$; this corresponds to the other irreducible 3-dimensional representation. One passes between these by the automorphism of A_4 given by conjugation by (45).

(c) Taking the inner product with the trivial representation gives

$$3+12+24cos(2\pi/5)+12+24cos(4\pi/5)-15=0 \qquad \text{or} \qquad cos(4\pi/5)+cos(2\pi/5)+\frac{1}{2}=0.$$

Using the double angle formula,

$$4cos^2(2\pi/5)+2cos(2\pi/5)-1=0.$$

We solve to get $cos(2\pi/5)=\frac{-1\pm\sqrt{5}}{4}$; matching sign to quadrant, we have

$$cos(2\pi/5)=\frac{-1+\sqrt{5}}{4} \qquad \text{and} \qquad cos(4\pi/5)=\frac{-1-\sqrt{5}}{4}.$$

8. Consider the ordered subset of $(\mathbb{Z}/2)^3$:

$$X=\{e_1,\ e_2,\ e_3,\ e_1+e_2,\ e_1+e_3,\ e_2+e_3,\ e_1+e_2+e_3\}.$$

Assign labels 1 through 7. We have the following permutations for each nontrivial class:

$$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\mapsto(12)(56),\qquad \begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}\mapsto(3657)(24),\qquad \begin{bmatrix}0&0&1\\1&0&1\\0&1&0\end{bmatrix}\mapsto(1234675),$$

$$\begin{bmatrix}0&0&1\\1&0&0\\0&1&1\end{bmatrix}\mapsto(1235746),\qquad \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\mapsto(123)(465).$$

The character values follow by counting fixed points. The sum of the squares of multiplicities of irreducible types is 2, and the trivial type occurs once.

$SL(3,\mathbb{Z}/2)$	e (1)	o2 (21)	o4 (42)	o7 (24)	o7i (24)	o3 (56)
π	7	3	1	0	0	1
π_6	6	2	0	-1	-1	0

9. (a)

$SL(3,\mathbb{Z}/2)$	e (1)	o2 (21)	o4 (42)	o7 (24)	o7i (24)	o3 (56)
χ_{triv}	1	1	1	1	1	1
π_6	6	2	0	-1	-1	0
$\pi_6 \otimes \pi_6$	36	4	0	1	1	0
$\chi_{\pi_6}(g^2)$	6	6	2	-1	-1	0
$\odot^2_{\pi_6}$	21	5	1	0	0	0
$\wedge^2\pi_6$	15	-1	-1	1	1	0

Checking inner products, the sum of squares of multiplicities for $\odot^2\pi_6$ is 6. The trivial type occurs once and π_6 occurs with multiplicity 2. What remains has character

$SL(3,\mathbb{Z}/2)$	e (1)	o2 (21)	o4 (42)	o7 (24)	o7i (24)	o3 (56)
χ_{triv}	1	1	1	1	1	1
π_6	6	2	0	-1	-1	0
$\odot^2_{\pi_6}$	21	5	1	0	0	0
π_8	8	0	0	1	1	-1

Checking inner products again, the sum of squares of multiplicities for $\wedge^2\pi_6$ is 2, so there are two irreducible types with multiplicity one. Again checking inner products of characters, we see that π_8 occurs with multiplicity one. What remains has character

$SL(3,\mathbb{Z}/2)$	e (1)	o2 (21)	o4 (42)	o7 (24)	o7i (24)	o3 (56)
χ_{triv}	1	1	1	1	1	1
π_6	6	2	0	-1	-1	0
π_8	8	0	0	1	1	-1
$\wedge^2\pi_6$	15	-1	-1	1	1	0
π_7	7	-1	-1	0	0	1

(b) The group order formula implies that the sum of the squares of the remaining dimensions is 18, so both remaining classes have dimension 3. Checking sums of squares of columns, we see that the remaining entries in the last column are zero. The remaining sum of moduli squared of the others are 2, 2, 4, 4. Taking the inner product of the sum of rows 2 and 3 with either of the last two rows gives $b = y = -1$. Again taking the inner product of either of the last two rows with row 4 gives $c = z = 1$.

Next $|d|^2 + |e|^2 = |u|^2 + |v|^2 = 4$. Since $d + e = u + v = -1$, each of d, e, u, v are non-real and $e = \overline{d} = u = \overline{v}$. Solving gives $d = \tau = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$.

$SL(3,\mathbb{Z}/2)$	e (1)	o2 (21)	o4 (42)	o7 (24)	o7i (24)	o3 (56)
χ_{triv}	1	1	1	1	1	1
π_6	6	2	0	-1	-1	0
π_8	8	0	0	1	1	-1
π_7	7	-1	-1	0	0	1
σ_1	3	-1	1	τ	$\overline{\tau}$	0
σ_2	3	-1	1	$\overline{\tau}$	τ	0

10. (a) We recall the conjugacy class structures of S_6 and S_5. We list by representative (cycle structure), number of elements, and group order.

$$|S_6| = 1 + 15 + 40 + 90 + 144 + 120 + 90 + 15 + 120 + 90 = 720.$$

(1) e, 1, 1,
(2) (12), 15, 2,
(3) (123), 40, 3,
(4) (1234), 90, 4,
(5) (12345), 144, 5,
(6) (123456), 120, 6,
(7) (12)(34), 90, 2,
(8) (12)(34)(56), 15, 2,
(9) (123)(45), 120, 6,
(10) (123)(456), 45, 3,
(11) (1234)(56), 90, 4.

$$|S_5| = 1 + 10 + 20 + 30 + 24 + 15 + 20 = 120$$

(1) e, 1, 1,
(2) (12), 10, 2,
(3) (123), 20, 3,
(4) (1234), 30, 4,
(5) (12345), 24, 5,
(6) (12)(34), 15, 2,
(7) (123)(45), 20, 6,

One notes that several classes in S_6 match by set order and group element order. Any subgroup isomorphic to S_5 must contain e and 5-cycles. If a transposition is also in this subgroup, these elements generate a standard S_5 or S_6. Thus the element of order 6 must be a 6-cycle; an element in the class (123)(45) produces a transposition upon cubing. The square of a 6-cycle is in the class for (123)(456) and its cube is in the class for (12)(34)(56). Finally the square of an order 4 element, in class (1234) or (1234)(56), is always in the class for (12)(34).

Counting fixed points, we have

$$6 + 24(1) + 20(0) + 20(0) + 10(0) + 15(2) = 60.$$

Thus the class for elements of order 4 have cycle structure (1234) to give the remaining 60 fixed points. One also sees this by considering the normalizer of the Sylow 5-subgroup $H = \langle(12345)\rangle$, which has 20 elements in both S_5 and S_6. Specifically $H = \langle(12345), (1243)\rangle$.

For the sum of squares formula,

$$36 + 24(1) + 20(0) + 20(0) + 10(0) + 15(4) + 30(4) = 240,$$

so there are two irreducible types with multiplicity one in the permutation representation on X.

(b) Identifying classes by cycle structure only, the trivial representation occurs with multiplicity one. Call the remaining type π'. We see immediately that $\pi' \cong sgn \otimes \pi_5$.

S_5	e (1)	$[2,2,2]$ (10)	$[3,3]$ (20)	$[4]$ (30)	$[5]$ (24)	$[2,2]$ (15)	$[6]$ (20)
π	6	0	0	2	1	2	0
π'	5	-1	-1	1	0	1	-1