INTRO TO GROUP THEORY - FEB. 8, 2012
PROBLEM SET 1 - GT1. DEFINITION OF A GROUP

1. Determine whether the following are groups of not. Explain. If not, find a subset that is a group with the given group multiplication.
   (1) $\mathbb{Z}$ under multipication,
   (2) $\mathbb{Z}/6$ under multiplication, and
   (3) the nonzero reals with multiplication $a \circ b = |ab|$.

2. For the modular integer group $\mathbb{Z}/12$ under addition, list all elements, and find their inverses and orders (smallest positive multiple to get a multiple of 12).

3. In the symmetric group S_4, compute
   (1) (12)(124)(12),
   (2) (124)(13)(142), and
   (3) (14)(13)(12).

4. Compute the number of elements in S_5. Find an element of order 6 in S_5.

5. If p is a prime, define $(\mathbb{Z}/p)^*$ to be group with elements $\{1, 2, ..., p-1\}$ using clockwork multiplication. Find the inverses for each element in $(\mathbb{Z}/5)^*$ and $(\mathbb{Z}/7)^*$.

6. Solve the following equation for y in S_3: $(12)y(123) = (13)$.

7. Let S^1 be the unit circle in the complex plane $\mathbb{C}$. That is, S^1 consists of complex numbers z with $|z| = 1$. Show that S^1 is a group under complex multiplication. (Hint: Euler's Formula and trig identities)

8. Let $M_2(\mathbb{R})$ be the set of 2×2 matrices with real entries. Matrix multiplication is defined as follows as a row-column dot product:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{pmatrix}.$$

Consider the subset $G = GL(2, \mathbb{R})$ of $M_2(\mathbb{R})$ consisting of (invertible) matrices with $det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0$.

(a) Show that $det(AB) = det(A)det(B)$. Explain why this shows that G is closed under multiplication.

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(b) Show that $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an identity element in G.

(c) For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in G and $det(A) = D$, show that

$$A^{-1} = \begin{pmatrix} d/D & -b/D \\ -c/D & a/D \end{pmatrix}.$$

(Rule: ignoring D, "switch and negate" - switch the diagonal entries and negate the off-diagonal entries.)

(d) Derive the formula in (c) by solving the equation $AB = I$ for B. What is $det(A^{-1})$? Is A^{-1} in G?

(e) Show that matrix multiplication is associative. (Tedious, but should be checked at least once.)

(f) Let $A = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$. Show that

(1) $AB \neq BA$,
(2) $(AB)^{-1} = B^{-1}A^{-1}$, and
(3) $(AB)^{-1} \neq A^{-1}B^{-1}$.

9. We can repeat the results in 8 by replacing $\mathbb{R}$ with $\mathbb{Z}/p$, where p is a prime. This gives the group $G = GL(2, \mathbb{Z}/p)$. With $p = 5$, find the inverse of $A = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$. Verify your answer. (Hint: $\frac{1}{a} = a^{-1}$.)