INTRO TO GROUP THEORY - APR. 18, 2012
PROBLEM SET 11
GT17/18. SYMMETRIC GROUPS, ETC.

1. (a) Find representatives for each conjugacy class of A_4, S_4, and S_5. Calculate the number of elements both using combinatorics and using centralizers.

(b) Verify the class equation for A_4, S_4, and S_5.

(c) Find the associated permutation matrix for each representative in (a) for S_5. Verify the trace formula in each case: the matrix trace equals the sum of the eigenvalues (with multiplicity).

2. (a) Explain why $Z(S_n) = Z(A_n) = \{e\}$ if $n \geq 4$.

(b) Verify the cycle structures associated to A_6 and S_6.

(c) How many k-cycles are in S_n?

(d) Show that S_n is isomorphic to a semidirect product of A_n and $\mathbb{Z}/2$.

3. (a) Suppose π is an action of the group G on the set X and σ is an automorphism of G. Show that $\pi \circ \sigma$ is also an action of G on X.

(b) if π is the usual action of S_3 on $X = \{1, 2, 3\}$ and σ is the inner automorphism by (12), describe $\pi \circ \sigma$.

4. The alternating group A_5 is isomorphic to the symmetry group I_{60} of rigid motions of a regular icosahedron.

(a) Describe all elements of both groups, and verify that the orders of elements match.

(b) Find or build an regular icosahedron. Identify 5 triplets of opposing edge pairs; in a given triplet, place parallel squares at each edge to form a cube. I_{60} permutes these 5 cubes, giving a homomorphism from I_{60} to S_5. Explain why this homomorphism is one-one. Below we note that A_5 is the unique subgroup of order 60 in S_5.

(c) Using the conjugacy classes in A_5, describe the conjugacy classes of I_{60}. Note that the classes with five elements contain inverse pairs.

(d) Show that $Aut(A_5) \cong S_5$ and $Out(A_5) \cong \mathbb{Z}/2$.

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5. (a) Show that S_n is generated by $\{(12),\ (23), \ldots, (n-1\ n)\}$.

(b) Show that S_n is generated by $(12\ldots n)$ and (12).

(c) If $n \neq 6$, show that every automorphism of S_n carries transpositions to transpositions. What happens when $n = 6$? (Hint: centralizers of order 2 elements)

(d) If $n \neq 6$, show that every automorphism of S_n carries $(12\ldots n)$ to another n-cycle. What happens when $n = 6$?

(e) If $n \neq 6, n > 2$, show that $Aut(S_n) = Inn(S_n) \cong S_n$.

(f) Show that A_n is the unique subgroup of order $n!/2$ in S_n.

(g) For $n \geq 4$, show that $Aut(S_n) \cong Aut(A_n)$. (Hint: do $n = 6$ separately)

(h) Calculate $Out(S_6)$.

6. (a) Verify the class equation for D_8 and D_{10}.

(b) Find all normal subgroups of D_{30} and D_{60}.

7. (a) Find all conjugacy classes for $G(12)$ described by

$$y^4 = e, \qquad x^3 = e, \qquad yxy^{-1} = x^2.$$

Find all normal subgroups of G.

(b) Find all conjugacy classes for $G(21)$ described by

$$y^3 = e, \qquad x^7 = e, \qquad yxy^{-1} = x^2.$$

Find all normal subgroups of G.

(c) Suppose p and q are primes with $q = kp + 1$. Suppose l is an element of order p in $(\mathbb{Z}/q)^*$. Find all conjugacy classes for G described by

$$y^p = e, \qquad x^q = e, \qquad yxy^{-1} = x^l.$$

Find all normal subgroups of G.

8. (a) Show that $Z(G)$ acts on the set of conjugacy classes of G by multiplication; that is, $zC_x = C_{zx}$. Describe for $G = Q,\ D_8,\ D_{4n},\ G(12)$.

(b) Show that $Out(G)$ acts on the set of conjugacy classes of G. Describe this action for $G = A_5,\ S_6,\ G(12),\ G(21)$.

9. (a) Suppose p is a prime, and let G be the subgroup of $SL(3, \mathbb{Z}/p)$ consisting of matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

where a, b, c are in $\mathbb{Z}/p$. Find the center of G.

(b) If $p = 3$, describe the class equation. General prime p?

10. (a) Explain why the following matrices in $SL(3, \mathbb{Z}/2)$ are not conjugate:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(b) Find the centralizer of each element in part (a) by solving $AX = XA$.

(c) Use (b) to show that $SL(3, \mathbb{Z}/2)$ is simple.