INTRO TO GROUP THEORY - APR. 25, 2012
PROBLEM SET 12
GT19/20. CAUCHY'S THEOREM/ SYLOW THEORY

1. (a) Find three non-abelian groups where $n < |G|$ divides $|G|$ but no element of G has order n.

(b) Repeat (a) with subgroups instead of elements.

2. Let p be a prime that divides $|G|$. Show that all Sylow p-subgroups H_p are isomorphic, that their normalizers $N(H_p)$ are isomorphic, and that $N(N(H_p)) = N(H_p)$. Also explain why the intersection of two normalizers cannot contain a Sylow p-subgroup of G.

3. Find all groups with exactly two or three conjugacy classes.

4. (a) Apply Sylow Theory to S_4, S_5, A_6, and S_6. Identify the isomorphism class of each Sylow p-subgroup and its normalizer.

(b) Show that a group of order 24 with no normal p-subgroups is isomorphic to S_4. (Hint: group action on Sylow 3-subgroups.)

(c) Find an example of a group of order 24 with 4 Sylow 3-subgroups and an element of order 6.

(d) Show that a group of order 24 without an element of order 6 must be isomorphic to S_4.

5. Show that all groups of orders 12, 30, 40, 42, and 45 must have a non-trivial, proper, normal subgroup.

6. (a) Apply Sylow Theory to D_{30} and D_{60}.

(b) Apply Sylow Theory to the dihedral groups D_{2n}. Find the normalizers of each Sylow p-subgroup.

7. (a) Apply Sylow Theory to a non-abelian group of order 21.

(b) Apply Sylow Theory to all groups of order pq where p and q are distinct primes. Find the normalizers of each Sylow p-subgroup.

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8. (a) Use the Corollary to Cayley's Theorem to show that all groups of orders 18, 24, 28, and 36 have a non-trivial, proper normal subgroup.

(b) Let $p < q$ be primes with $p^k < q$. Show that every group of order $p^i q^j$ has a normal subgroup if $1 \leq i \leq k, j \geq 0$. (In fact, Burnside's Theorem says that any group of order $p^i q^j$ with $i, j > 0$ has a non-trivial, proper, normal subgroup.)

9. (a) Show that every group of order < 60 has a non-trivial, proper, normal subgroup.

(b) Every group with order between 60 and 168 has a non-trivial, proper, normal subgroup. Attempt to prove. For what cases should be invoke Burnside's Theorem? What cases remain?

10. Let q be a prime. Explain why $G = SL(2, \mathbb{Z}/q)$ must have elements of order 2, 3 and q. Find one of each. How many Sylow q-subgroups are there?

(b) Apply Sylow Theory when $q = 3, 5, 7$.

11. We outline the Sylow Theory for any simple group G of order 168. Recall by the Corollary to Cayley's Theorem that there are no subgroups of order 28, 42, 56, or 84.

(a) Calculate the number of Sylow 7-subgroups and their normalizers. How many elements of order 7 are there? How many conjugacy classes with order 7 elements? Explain why there are no elements of order 14 or 21. (Hint: briefly detour to Sylow 3-subgroups to find the normalizers.)

(b) Calculate the number of Sylow 3-subgroups and their normalizers. How many elements of order 3 are there? How many conjugacy classes with order 3 elements? Explain why there are no elements of order 6, 12, or 24. (Hint: consider Sylow 3-subgroups in the intersection of two Sylow 7-subgroup normalizers.)

(c) Calculate the number of Sylow 2-subgroups and their normalizers. How many elements of order 4 and 2 are there? How many conjugacy classes with order 4 and 2 elements? Explain why there are no elements of order 8. (Hint: every group of order 8 has non-trivial center.)

(d) Compare with the results from before for $SL(3, \mathbb{Z}/2)$. Find a subgroup of order 24 in $SL(3, \mathbb{Z}/2)$ (isomorphic to S_4).